The Open University

MS221
Exploring Mathematics

Exercise Book B

Contents

The exercises in this booklet are intended to give further practice, should you require it, in handling the main mathematical ideas in each chapter of MS221, Block B. The exercises are ordered by chapter and section, and are numbered correspondingly: for example, Exercise 3.2 for Chapter B1 is the second exercise on Section 3 of that chapter.

Exercises for Chapter B1

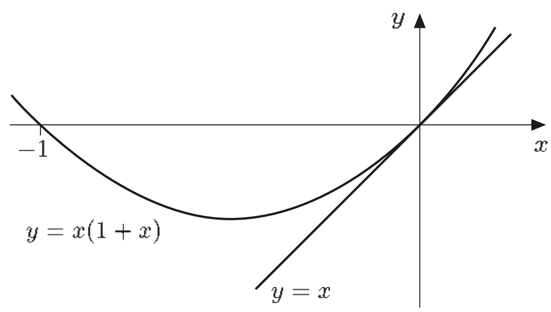
Section 1

Exercise 1.1

For each of the following iteration sequences, calculate the first five terms of the sequence (correct to three significant figures) and construct these terms, where possible, using the graph provided.

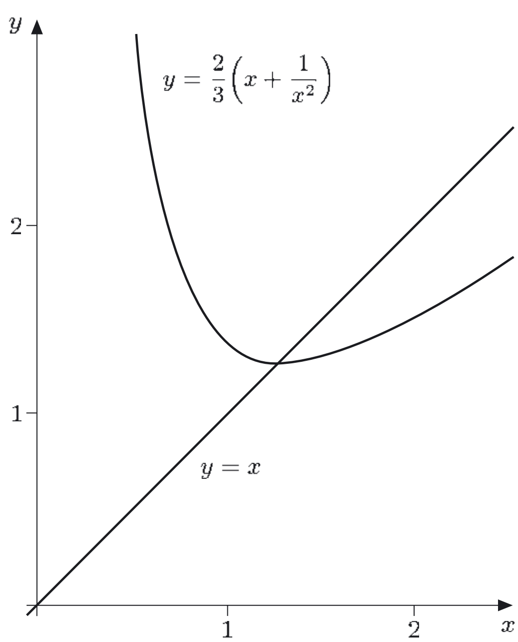
(a)

$$x_0 = -0.5, \quad x_{n+1} = x_n(1 + x_n) \quad (n = 0, 1, 2, \ldots)$$



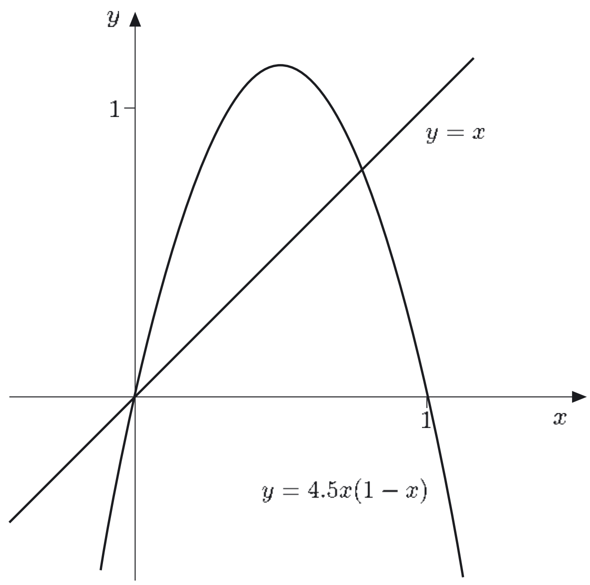
(b)

$$x_0 = 0.6, \quad x_{n+1} = \tfrac{2}{3}\left(x_n + 1/x_n^2\right) \quad (n = 0, 1, 2, \ldots)$$



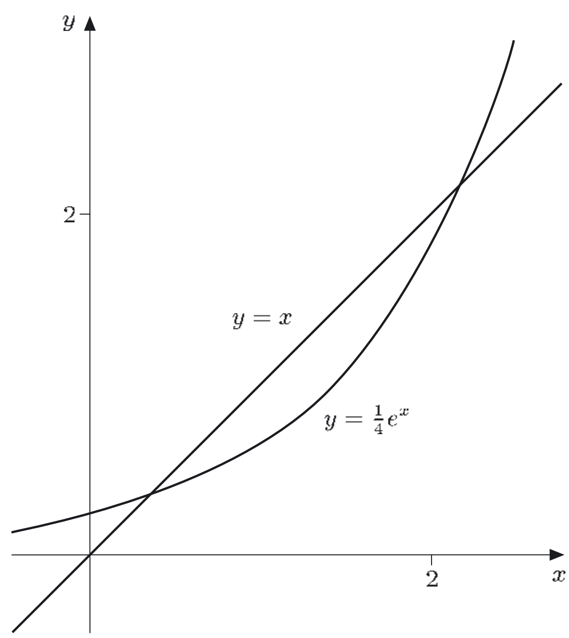
(c)

$$x_0 = 0.75, \quad x_{n+1} = 4.5x_n(1 - x_n) \quad (n = 0, 1, 2, \ldots)$$



(d)

$$x_0 = 2, \quad x_{n+1} = \tfrac{1}{4}e^{x_n} \quad (n = 0, 1, 2, \ldots)$$



Exercise 1.2

Given that the iteration sequences in Exercise 1.1(a) and (b) are both convergent, find the limit of each of these sequences.

Exercise 1.3

Given that the iteration sequence in Exercise 1.1(d) is convergent, what can you say about the limit of this sequence?

Exercise 1.4

Describe the long-term behaviour of the iteration sequence in Exercise 1.1(c).

Exercise 1.5

(a) Sketch, on the same axes, the graphs of $y = x$ and $y = f(x)$, where

$$f(x) = x^2 + x - \tfrac{3}{4}.$$

(b) Determine the fixed points of the function f.

(c) Use graphical iteration to describe the long-term behaviour of iteration sequences given by

$$x_{n+1} = x_n^2 + x_n - \tfrac{3}{4} \quad (n = 0, 1, 2, \ldots),$$

for each of the following initial terms.

(i) $x_0 = -1.25$

(ii) $x_0 = 1$

Section 2

Exercise 2.1

Use the quadratic gradient formula in Frame 4 of Section 2 to find the gradient at the given point on the graph of $y = f(x)$ in each of the following cases.

(a) $f(x) = 3x^2, \quad (-\tfrac{1}{3}, \tfrac{1}{3})$

(b) $f(x) = 3x^2 - 1, \quad (-\tfrac{1}{3}, -\tfrac{2}{3})$

(c) $f(x) = -x^2 + 2x - 1, \quad (3, -4)$

(d) $f(x) = \tfrac{1}{5}x^2 - \tfrac{1}{2}x + 1, \quad (1, \tfrac{7}{10})$.

Exercise 2.2

For each of the following functions f, find and classify the fixed points of f.

(a) $f(x) = x^2 + 2x$

(b) $f(x) = -4.5x^2 + 4.5x$

(c) $f(x) = x^2 + x + \tfrac{3}{4}$

(d) $f(x) = x^2 + x - \tfrac{3}{4}$

(e) $f(x) = -\tfrac{1}{2}x^2 + \tfrac{1}{2}x + 2$

Exercise 2.3

(a) Sketch, on the same axes, the graphs of $y = x$ and $y = f(x)$, where

$$f(x) = \tfrac{1}{2}x^2 + \tfrac{1}{5}.$$

(b) Find and classify the fixed points of f.

(c) One of the fixed points of f is attracting.

(i) Use the graphical criterion to find an interval of attraction I for this fixed point.

(ii) Use the gradient criterion to find an interval of attraction J for this fixed point.

(d) Describe the long-term behaviour of iteration sequences given by

$$x_{n+1} = f(x_n) \quad (n = 0, 1, 2, \ldots),$$

for each of the following initial terms.

(i) $x_0 = 0.5$

(ii) $x_0 = -0.5$

(iii) $x_0 = 1$

(iv) $x_0 = 2$

(v) $x_0 = -1$

(vi) $x_0 = 1 + \tfrac{1}{5}\sqrt{15}$

Exercise 2.4

The function

$$f(x) = \tfrac{1}{2}x^2 + x - 1$$

has a fixed point at $-\sqrt{2}$.

(a) Show that this fixed point is attracting.

(b) Use the gradient criterion to find an interval of attraction I for this fixed point.

(c) Describe the long-term behaviour of iteration sequences given by

$$x_{n+1} = f(x_n) \quad (n = 0, 1, 2, \ldots),$$

for each of the following initial terms.

(i) $x_0 = -1$

(ii) $x_0 = 1$

(iii) $x_0 = \sqrt{2}$

(iv) $x_0 = 2$

Section 3

Exercise 3.1

For each of the following functions f and g, show that the composite function $g \circ f$ can be formed and describe this composite function using two-line notation.

(a) $f(x) = \sin x \quad (x \in [2\pi, 3\pi])$ and $g(x) = \sqrt{x}$

(b) $f(x) = \sin x$ and $g(x) = \sqrt{x+3}$

Exercise 3.2

For each of the following functions f, show that the given values a and b form a 2-cycle of f and classify this 2-cycle.

(a) $f(x) = \frac{7}{2}x(1-x), \quad a = \frac{3}{7}, \quad b = \frac{6}{7}$.

(b) $f(x) = x^2 - \frac{7}{9}, \quad a = -\frac{2}{3}, \quad b = -\frac{1}{3}$.

Section 5

Exercise 5.1

(a) How many five-digit permutations can be formed using the digits 1, 2, 3, 4, 5?

(b) How many three-digit permutations can be formed from the digits from the list 1, 2, 3, 4, 5?

(c) How many five-digit combinations can be selected from the digits 1, 2, 3, 4, 5, 6, 7?

Exercise 5.2

(a) In how many ways can the twelve letters A, B, C, D, E, F, G, H, I, J, K, L be arranged?

(b) How many six-letter arrangements can be made using letters from the list A, B, C, D, E, F, G, H, I, J, K, L at most once each?

(c) How many seven-letter arrangements beginning with C can be made using letters from the list A, B, C, D, E, F, G, H, I, J, K, L at most once each?

(d) In how many ways can six letters be selected from the list A, B, C, D, E, F, G, H, I, J, K, L?

Exercise 5.3

Find the first four terms in the expansion of each of the following expressions.

(a) $(1 + 3x)^7$

(b) $(3 - x^2)^6$

Exercise 5.4

Find the coefficient of x^8 in the expansion of each of the following expressions.

(a) $(1 + x)^{10}$

(b) $(x^2 - 2)^{10}$

(c) $(2 + x^3)^{11}$

Exercise 5.5

(a) Find the coefficient of p^7q^3 in the expansion of $(p + q)^{10}$.

(b) Find the coefficient of x^5y^7 in the expansion of $(\frac{1}{4}x - 2y)^{12}$.

(c) Find the constant term in the expansion of $\left(x + \dfrac{1}{x^2}\right)^9$.

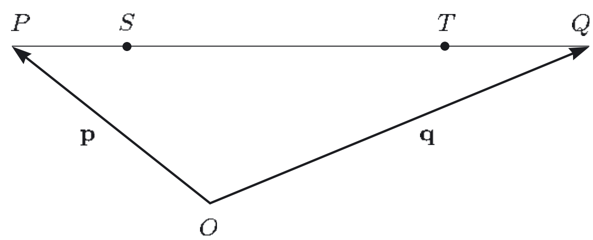
(d) Find the constant term in the expansion of $\left(x + \dfrac{1}{x^2}\right)^8$.

Exercises for Chapter B2

Section 1

Exercise 1.1

(a)



Find the position vectors (in terms of $\mathbf{p}$ and $\mathbf{q}$, the position vectors of P and Q) of the points S and T, where S lies a fifth of the way from P to Q, and T lies a quarter of the way from Q to P.

(b) Determine S and T in the particular case where $P = (3, -1)$ and $Q = (-7, 11)$.

Exercise 1.2

Determine the matrix that represents $r_{2\pi/3}$, the rotation of the plane about the origin through the angle $2\pi/3$ anticlockwise. Use the matrix to find the image under this rotation of the triangle with vertices at $(2, 0)$, $(0, 1)$, $(1, 2)$.

Exercise 1.3

Determine the matrix that represents $q_{\pi/8}$, the reflection of the plane in a line that passes through the origin and makes an angle $\pi/8$ measured anticlockwise from the positive x-axis. Use the matrix to find the image under this reflection of the square with vertices at $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$.

Exercise 1.4

Identify each of the following matrices as a rotation matrix $\mathbf{R}_\theta$ or a reflection matrix $\mathbf{Q}_\theta$, and determine the angle θ in each case.

(a) $\begin{pmatrix} -\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}$ (b) $\begin{pmatrix} -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{pmatrix}$

Section 2

Exercise 2.1

For each of the following transformations f, either explain why f is not linear, or write down the matrix that represents f.

(a) $f(x, y) = (0, y - 3x)$.

(b) $f(x, y) = (x - y - 1, 2x + y)$.

(c) $f(x, y) = (x, y)$.

(d) f is the translation $t_{-1,1}$.

(e) f reflects the plane in the line $y = x$.

(f) f rotates the plane anticlockwise through $\frac{1}{2}\pi$ about the point $(1, 0)$.

Exercise 2.2

For each of the following linear transformations f, write down the matrix that represents the transformation.

(a) f scales the plane by a factor 4 in the direction of the x-axis and by a factor -2 in the direction of the y-axis.

(b) f reflects the plane in a line through the origin that makes an angle $-\pi/6$ with the positive x-axis.

(c) f shears the plane parallel to the x-axis in such a way that points at a distance 1 below the x-axis shift 3 units to the left.

(d) f maps the points $(1, 0)$ and $(0, 1)$ to the points $(-1, -2)$ and $(3, 0)$, respectively.

Exercise 2.3

For each of the following linear transformations f represented by a matrix, draw a sketch showing the effect of f on the unit grid, and hence give a geometric interpretation of f. In addition, describe what effect f has on areas and orientation.

(a) f represented by the matrix

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -2 \end{pmatrix}.$$

(b) f represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}.$$

Exercise 2.4

Let f be the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} -2 & 0 \\ 4 & -3 \end{pmatrix}.$$

Draw a sketch showing the effect of f on the unit grid and describe what effect f has on areas and orientation.

Exercise 2.5

Identify the type of the linear transformation f represented by each of the matrices $\mathbf{A}$ given below. In each case, describe briefly the geometric effect of f. Also, in each case, calculate the factor by which areas are scaled and state whether orientation is preserved.

(a) $\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} -6 & 4 \\ 9 & -6 \end{pmatrix}$

Exercise 2.6

Let f be the linear transformation that sends $(1, 0)$ to $(4, 2)$ and $(0, 1)$ to $(-2, -7)$. Write down the matrix that represents f and use it to calculate the area of the triangle T with vertices at $(0, 0)$, $(4, 2)$, $(-2, -7)$.

Exercise 2.7

Let f be the linear transformation represented by the matrix

$$\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}.$$

Show, without using any results from Section 3 of the chapter, that f is (a) one-one; (b) onto.

Section 3

Exercise 3.1

(a) Write down the matrices $\mathbf{Q}_{\pi/8}$ and $\mathbf{R}_0$ that represent the linear transformations, $q_{\pi/8}$ and r_0, respectively.

(b) Verify, using matrix multiplication, that $\mathbf{Q}_{\pi/8}\mathbf{Q}_{\pi/8} = \mathbf{R}_0$.

Exercise 3.2

Let $f\colon \mathbf{x} \longmapsto \mathbf{Ax}$ and $g\colon \mathbf{x} \longmapsto \mathbf{Bx}$ be linear transformations, where

$$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 3 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -1 & 2 \\ 4 & 3 \end{pmatrix}.$$

Determine the matrices that represent each of the following composite transformations.

(a) $g \circ f$ (b) $f \circ g$

Exercise 3.3

Let $f\colon \mathbf{x} \longmapsto \mathbf{Ax}$, $g\colon \mathbf{x} \longmapsto \mathbf{Bx}$ and $h\colon \mathbf{x} \longmapsto \mathbf{Cx}$ be linear transformations represented by the matrices

$$\mathbf{A} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} 4 & -6 \\ 0 & 3 \end{pmatrix}.$$

(a) Describe the geometric effect of each of f and g.

(b) Show that $h = g \circ f$.

Exercise 3.4

Let f be a linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 8 & 5 \end{pmatrix}.$$

(a) Show that f is one-one and onto.

(b) Determine f^{-1}.

(c) Find the point (x, y) such that $f(x, y) = (1, 3)$.

Exercise 3.5

For each of the following matrices $\mathbf{A}$, decide whether $\mathbf{A}$ is invertible. For those that are invertible, calculate $\mathbf{A}^{-1}$.

(a) $\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 \\ 3 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 1 \\ -6 & -2 \end{pmatrix}$

Exercise 3.6

Let f be the linear transformation that maps $(1, 0)$ to $(-1, 2)$ and $(0, 1)$ to $(2, -3)$. Also let g be the linear transformation that maps $(1, 0)$ to $(3, 4)$ and $(0, 1)$ to $(1, -1)$.

(a) Write down the matrices $\mathbf{A}$ and $\mathbf{B}$ that represent f and g, respectively.

(b) Use the matrix that represents f to find a linear transformation that maps $(-1, 2)$ back to $(1, 0)$ and $(2, -3)$ back to $(0, 1)$.

(c) Find a linear transformation that maps $(-1, 2)$ to $(3, 4)$ and $(2, -3)$ to $(1, -1)$.

Exercise 3.7

Let f be the linear transformation represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Find the equation of the image $f(\mathscr{C})$ of the unit circle $\mathscr{C}$ under f, and calculate the area enclosed by the image.

Section 4

Exercise 4.1

(a) Find the affine transformation that sends the points $(0, 0)$, $(1, 0)$, $(0, 1)$ to the points $(-1, -3)$, $(7, 5)$, $(3, 0)$, respectively.

(b) Hence find the area of the triangle T with vertices at $(-1, -3)$, $(7, 5)$, $(3, 0)$.

Exercise 4.2

Find the affine transformation that describes an anticlockwise rotation about the point $(4, 2)$ through $\pi/3$ radians.

Exercise 4.3

(a) Determine the matrix that represents a reflection in the y-axis.

(b) By first translating the point $(3, 0)$ to the origin, find the affine transformation that describes a reflection in the line $x = 3$.

Exercises for Chapter B3

Section 1

Exercise 1.1

Describe the fixed points and invariant lines through the origin of each of the following linear transformations.

(a) $r_{-2\pi}$

(b) $q_{\pi/3}$

(c) scaling with factors -4 and 1

(d) y-shear with factor -1.

Exercise 1.2

This exercise concerns the reflection $q_{-\pi/4}$.

(a) Use the matrix $\mathbf{Q}_{-\pi/4}$ representing this reflection to check that every point on the line $y = -x$ is a fixed point of this reflection.

(b) Show that the x-axis is *not* an invariant line of this reflection.

Section 2

Exercise 2.1

Let $\mathbf{A}$ be the matrix $\begin{pmatrix} -\frac{4}{3} & \frac{14}{3} \\ \frac{10}{3} & -\frac{8}{3} \end{pmatrix}$.

(a) Given that 2 is an eigenvalue of $\mathbf{A}$, find the equation of the corresponding eigenline and write down two eigenvectors for this eigenline.

(b) Given that $y = -x$ is an eigenline of $\mathbf{A}$, find the corresponding eigenvalue.

Exercise 2.2

Let $\mathbf{A}$ be the matrix $\begin{pmatrix} \frac{2}{5} & -\frac{36}{5} \\ -\frac{6}{5} & \frac{8}{5} \end{pmatrix}$.

(a) Show that $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$, and find the corresponding eigenvalue and equation of the corresponding eigenline.

(b) Decide whether 2 is an eigenvalue of $\mathbf{A}$.

(c) Decide whether $y = \frac{1}{3}x$ is an eigenline of $\mathbf{A}$.

Exercise 2.3

Find the eigenvalues (if they exist) and eigenlines of each of the following matrices. For each eigenline, give one eigenvector.

(a) $\begin{pmatrix} 2 & 6 \\ 2 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & -1 \\ -\frac{2}{3} & \frac{4}{3} \end{pmatrix}$

Section 3

Exercise 3.1

The matrix $\mathbf{A}$ is given by $\mathbf{A} = \mathbf{PDP}^{-1}$, where $\mathbf{P} = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$.

(a) Write down the eigenvectors of $\mathbf{A}$, and the corresponding eigenlines.

(b) Find $\mathbf{A}^3$ (without using direct multiplication).

Exercise 3.2

Let $\mathbf{A}$ be the matrix $\begin{pmatrix} 8 & 3 \\ -18 & -7 \end{pmatrix}$.

(a) Check by direct calculation that $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ for the eigenvalue 2 and that $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ is an eigenvector of $\mathbf{A}$ for the eigenvalue -1.

(b) Taking $\mathbf{D}$ to be the matrix $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, the eigenvector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ for the eigenvalue 2 and the eigenvector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ for the eigenvalue -1 in Steps 2 and 4 of the strategy in Chapter B3, Section 3, to express $\mathbf{A}$ in the form $\mathbf{PDP}^{-1}$, find the corresponding matrices $\mathbf{P}$ and $\mathbf{P}^{-1}$ which arise from Steps 5 and 6 of the strategy. Check by multiplying out the matrix product that $\mathbf{PDP}^{-1} = \mathbf{A}$.

(c) Taking $\mathbf{D}$ to be the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, the eigenvector $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ for the eigenvalue -1 and the eigenvector $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ for the eigenvalue 2 in Steps 2 and 4 of the strategy in Chapter B3, Section 3, to express $\mathbf{A}$ in the form $\mathbf{PDP}^{-1}$, find the corresponding matrices $\mathbf{P}$ and $\mathbf{P}^{-1}$ which arise from Steps 5 and 6 of the strategy. Check by multiplying out the matrix product that $\mathbf{A} = \mathbf{PDP}^{-1}$.

(d) Find $\mathbf{A}^4$ (without using direct multiplication).

Exercise 3.3

Express each of the following matrices in the form $\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix. (You can use your solutions to Exercise 2.3 here.)

(a) $\begin{pmatrix} 2 & 6 \\ 2 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -1 \\ -\frac{2}{3} & \frac{4}{3} \end{pmatrix}$

Section 4

Exercise 4.1

The matrix $\mathbf{A} = \begin{pmatrix} \frac{9}{2} & -\frac{5}{4} \\ 10 & -3 \end{pmatrix}$ equals $\mathbf{PDP}^{-1}$, where $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$. Determine the matrix $\mathbf{A}^n$.

Exercise 4.2

The matrix $\mathbf{A} = \begin{pmatrix} 1 & 6 \\ -2 & -6 \end{pmatrix}$ has eigenvalues -2 and -3 with corresponding eigenlines $y = -\frac{1}{2}x$ and $y = -\frac{2}{3}x$. Describe in words the long-term behaviour of the iteration sequences (x_n, y_n) generated by $\mathbf{A}$ with each of the following initial points.

(a) $(-6, 4)$

(b) $(-8, -8)$

Exercise 4.3

The matrix $\mathbf{A} = \begin{pmatrix} 5 & 9 \\ -3 & -7 \end{pmatrix}$ equals $\mathbf{PDP}^{-1}$, where $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$.

(a) (i) Check by direct calculation that $\mathbf{A} = \mathbf{PDP}^{-1}$.

(ii) Determine the matrix $\mathbf{A}^n$.

(b) Write down the eigenvalues and corresponding eigenlines of $\mathbf{A}$.

(c) Describe in words the long-term behaviour of the iteration sequences (x_n, y_n) generated by $\mathbf{A}$ with each of the following initial points.

(i) $(-3, 1)$

(ii) $(0, 0)$

(iii) $(4, -2)$

(iv) $(-3, 3)$

Section 5

Exercise 5.1

The matrix $\mathbf{A} = \begin{pmatrix} 3 & 5 \\ -\frac{5}{3} & -\frac{17}{6} \end{pmatrix}$ has eigenvalues $\frac{1}{2}$ and $-\frac{1}{3}$ with corresponding eigenlines $y = -\frac{1}{2}x$ and $y = -\frac{2}{3}x$. Describe in words the long-term behaviour of the iteration sequences (x_n, y_n) generated by $\mathbf{A}$ with each of the following initial points.

(a) $(-6, 3)$

(b) $(5, 4)$

Exercise 5.2

The matrix $\mathbf{A} = \begin{pmatrix} \frac{9}{2} & -\frac{5}{4} \\ 10 & -3 \end{pmatrix}$ equals $\mathbf{PDP}^{-1}$, where $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$.

(a) Write down the eigenvalues and corresponding eigenlines of $\mathbf{A}$.

(b) Describe in words the long-term behaviour of the iteration sequences (x_n, y_n) generated by $\mathbf{A}$ with each of the following initial points.

(i) $(3, 12)$

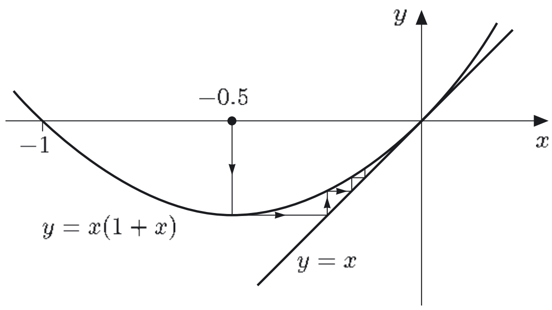
(ii) $(2, -2)$

Solutions for Chapter B1

Solution 1.1

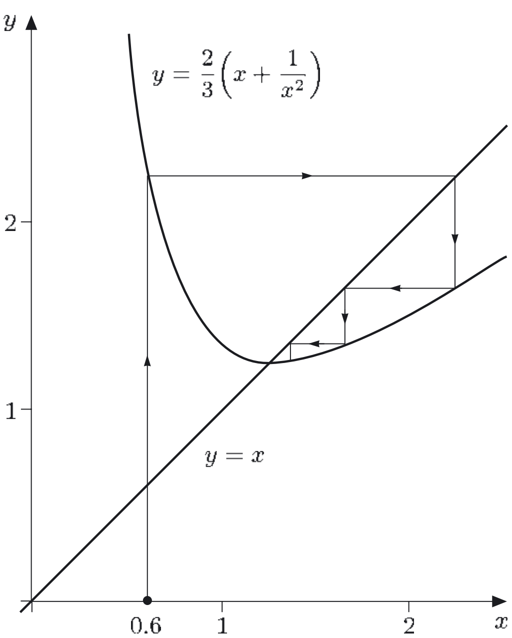
(a) The first five terms are (to 3 s.f.):

$$-0.5,\ -0.25,\ -0.188,\ -0.152,\ -0.129.$$



(b) The first five terms are (to 3 s.f.):

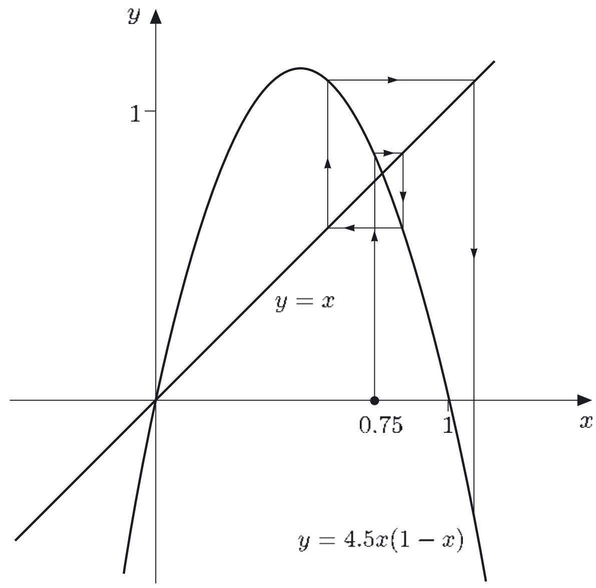
$$0.6,\ 2.25,\ 1.63,\ 1.34,\ 1.26.$$



(c) The first five terms are (to 3 s.f.):

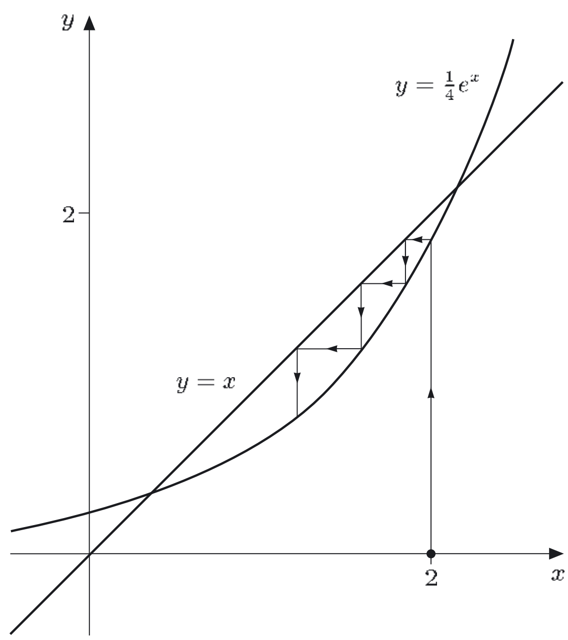
$$0.75,\ 0.844,\ 0.593,\ 1.09,\ -0.420.$$

This is illustrated in the following graph.



(d) The first five terms are (to 3 s.f.):

$$2,\ 1.85,\ 1.59,\ 1.22,\ 0.847.$$



Solution 1.2

(a) By the Fixed Point Rule, the limit is a fixed point of the function $f(x) = x(1+x)$. The fixed point equation is

$$x(1+x) = x,$$

which can be rearranged as

$$x^2 = 0.$$

Thus the only fixed point is 0, so this must be the limit of the sequence.

(b) By the Fixed Point Rule, the limit is a fixed point of the function $f(x) = \frac{2}{3}(x + 1/x^2)$. The fixed point equation is

$$\tfrac{2}{3}(x + 1/x^2) = x,$$

which can be multiplied through by $\frac{3}{2}$ to obtain $x + 1/x^2 = \frac{3}{2}x$ and then rearranged as

$$1/x^2 = \tfrac{1}{2}x; \quad \text{that is,} \quad x^3 = 2.$$

Thus the only fixed point is $2^{1/3}$.

Solution 1.3

By the Fixed Point Rule, the limit is a fixed point of the function $f(x) = \frac{1}{4}e^x$; that is, it is a solution of the equation

$$e^x = 4x.$$

Also, the limit lies below 0.847.

(There is no simple formula for solving the equation $e^x = 4x$, but further calculation of this sequence x_n shows that the solution is approximately 0.357. There is one further solution to the equation, approximately 2.15.)

Solution 1.4

The sixth term, x_5, of the sequence is -2.68 (to 3 s.f.). By continuing the graphical iteration in the diagram in Solution 1.1(c), it can be seen that

$$x_n \to -\infty \text{ as } n \to \infty.$$

Solution 1.5

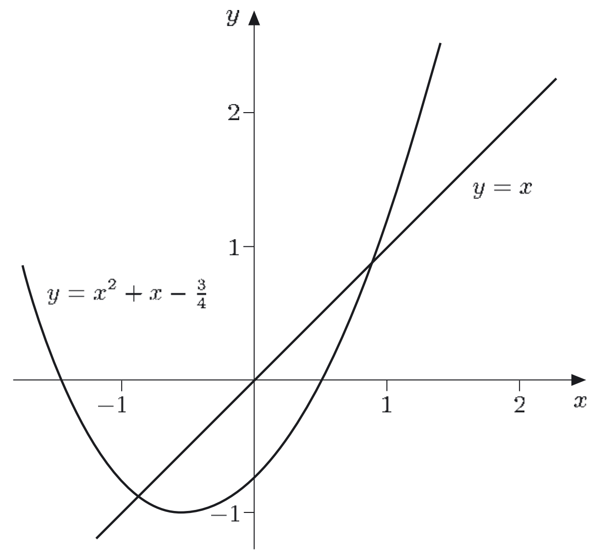
(a) First we write $f(x)$ in completed-square form:

$$\begin{aligned} f(x) &= x^2 + x - \tfrac{3}{4} \\ &= (x + \tfrac{1}{2})^2 - \tfrac{1}{4} - \tfrac{3}{4} \\ &= (x + \tfrac{1}{2})^2 - 1. \end{aligned}$$

Therefore, the graph of $y = f(x)$ can be obtained from the graph of $y = x^2$ by performing:

- ◇ a horizontal translation by $\frac{1}{2}$ of a unit to the left;
- ◇ a vertical translation by 1 unit downwards.

The y-intercept is $f(0) = -\frac{3}{4}$ and the x-intercepts are the solutions of the equation $f(x) = 0$; that is, $\frac{1}{2}(-1 \pm \sqrt{4})$, which simplify to -1.5 and 0.5. Thus the graph of $y = f(x)$ is as follows.

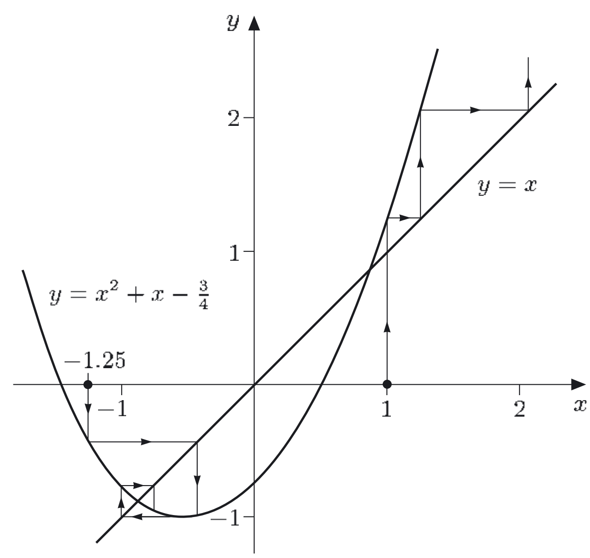


(b) The fixed point equation is

$$x^2 + x - \tfrac{3}{4} = x; \quad \text{that is,} \quad x^2 - \tfrac{3}{4} = 0,$$

which has solutions $\pm\frac{1}{2}\sqrt{3}$ ($\simeq \pm 0.866$).

(c) The effect of graphical iteration with the two initial terms $x_0 = -1.25$ and $x_0 = 1$ is as follows.



(i) If $x_0 = -1.25$, then $x_n \to -\frac{1}{2}\sqrt{3}$ as $n \to \infty$.

(ii) If $x_0 = 1$, then $x_n \to \infty$ as $n \to \infty$.

Solution 2.1

(a) If $f(x) = 3x^2$, then $f'(x) = 6x$. Thus the gradient at the point $(-\frac{1}{3}, \frac{1}{3})$ is

$$f'\left(-\tfrac{1}{3}\right) = 6 \times \left(-\tfrac{1}{3}\right) = -2.$$

(b) If $f(x) = 3x^2 - 1$, then $f'(x) = 6x$. Thus the gradient at the point $(-\frac{1}{3}, -\frac{2}{3})$ is

$$f'\left(-\tfrac{1}{3}\right) = 6 \times \left(-\tfrac{1}{3}\right) = -2.$$

(c) If $f(x) = -x^2 + 2x - 1$, then $f'(x) = -2x + 2$. Thus the gradient at the point $(3, -4)$ is

$$f'(3) = -2 \times 3 + 2 = -4.$$

(d) If $f(x) = \frac{1}{5}x^2 - \frac{1}{2}x + 1$, then $f'(x) = \frac{2}{5}x - \frac{1}{2}$. Thus the gradient at the point $(1, \frac{7}{10})$ is

$$f'(1) = \tfrac{2}{5} \times 1 - \tfrac{1}{2} = -\tfrac{1}{10}.$$

Solution 2.2

(a) For $f(x) = x^2 + 2x$, the fixed point equation is

$$x^2 + 2x = x; \quad \text{that is,} \quad x^2 + x = 0,$$

which has solutions $x = -1$ and $x = 0$.

Now $f'(x) = 2x + 2$, and hence

$$f'(-1) = 0,$$
$$f'(0) = 2 > 1.$$

Thus 0 is a repelling fixed point and -1 is a super-attracting fixed point.

(b) For $f(x) = -4.5x^2 + 4.5x$, the fixed point equation is

$$-4.5x^2 + 4.5x = x; \quad \text{that is,} \quad 4.5x^2 - 3.5x = 0,$$

which has solutions $x = 0$ and $x = \frac{7}{9}$.

Now $f'(x) = -9x + 4.5$, and hence

$$f'(0) = 4.5 > 1,$$
$$f'\left(\tfrac{7}{9}\right) = -2.5 < -1.$$

Thus both 0 and $\frac{7}{9}$ are repelling fixed points. (This function is the same as that dealt with in Exercises 1.1(c) and 1.4.)

(c) For $f(x) = x^2 + x + \frac{3}{4}$, the fixed point equation is

$$x^2 + x + \tfrac{3}{4} = x; \quad \text{that is,} \quad x^2 + \tfrac{3}{4} = 0,$$

which has no solutions. So there are no fixed points to classify.

(d) For $f(x) = x^2 + x - \frac{3}{4}$, we know from Solution 1.5(b) that the fixed points are $\pm\frac{1}{2}\sqrt{3}$.

Now $f'(x) = 2x + 1$, and hence

$$\begin{aligned} f'\left(-\tfrac{1}{2}\sqrt{3}\right) &= 2 \times \left(-\tfrac{1}{2}\sqrt{3}\right) + 1 = 1 - \sqrt{3} \\ &\simeq -0.732 > -1, \\ f'\left(\tfrac{1}{2}\sqrt{3}\right) &= 2 \times \left(\tfrac{1}{2}\sqrt{3}\right) + 1 = 1 + \sqrt{3} \\ &\simeq 2.732 > 1. \end{aligned}$$

Thus $-\frac{1}{2}\sqrt{3}$ is an attracting fixed point, whereas $\frac{1}{2}\sqrt{3}$ is a repelling fixed point; see the diagram in Solution 1.5(a). (This function is the same as that dealt with in Exercise 1.5.)

(e) For $f(x) = -\frac{1}{2}x^2 + \frac{1}{2}x + 2$, the fixed point equation is

$$-\tfrac{1}{2}x^2 + \tfrac{1}{2}x + 2 = x; \quad \text{that is,} \quad \tfrac{1}{2}x^2 + \tfrac{1}{2}x - 2 = 0,$$

which has solutions $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{17}$.

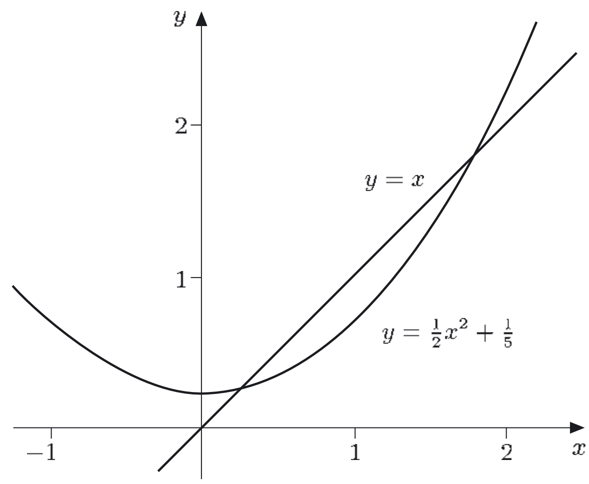
Now $f'(x) = -x + \frac{1}{2}$, and hence

$$\begin{aligned} f'\left(-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{17}\right) &= -\left(-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{17}\right) + \tfrac{1}{2} \\ &= 1 + \tfrac{1}{2}\sqrt{17} \\ &\simeq 3.062 > 1, \\ f'\left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{17}\right) &= -\left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{17}\right) + \tfrac{1}{2} \\ &= 1 - \tfrac{1}{2}\sqrt{17} \\ &\simeq -1.062 < -1. \end{aligned}$$

Thus both fixed points are repelling.

Solution 2.3

(a)



(b) For $f(x) = \frac{1}{2}x^2 + \frac{1}{5}$, the fixed point equation is

$$\tfrac{1}{2}x^2 + \tfrac{1}{5} = x; \quad \text{that is,} \quad \tfrac{1}{2}x^2 - x + \tfrac{1}{5} = 0,$$

which has solutions $x = 1 - \frac{1}{5}\sqrt{15} \simeq 0.225$ and $x = 1 + \frac{1}{5}\sqrt{15} \simeq 1.775$.

Now $f'(x) = x$, and hence

$$\begin{aligned} f'\left(1 - \tfrac{1}{5}\sqrt{15}\right) &= 1 - \tfrac{1}{5}\sqrt{15} \simeq 0.225 < 1, \\ f'\left(1 + \tfrac{1}{5}\sqrt{15}\right) &= 1 + \tfrac{1}{5}\sqrt{15} \simeq 1.775 > 1. \end{aligned}$$

Thus $1 - \frac{1}{5}\sqrt{15}$ is an attracting fixed point and $1 + \frac{1}{5}\sqrt{15}$ is a repelling fixed point.

(c) (i) From the graph in part (a), the function f is increasing on the open interval $I = \left(0, 1 + \frac{1}{5}\sqrt{15}\right)$ and $1 - \frac{1}{5}\sqrt{15}$ is the only fixed point of f in I. Thus by the graphical criterion in Section 2, I is an interval of attraction for the fixed point $1 - \frac{1}{5}\sqrt{15}$.

(ii) As $f'(x) = x$, the condition $|f'(x)| < 1$ can be written as the two inequalities

$$-1 < x < 1.$$

The attracting fixed point $1 - \frac{1}{5}\sqrt{15} \simeq 0.225$ is nearer to 1 than -1, so we choose J to have right-hand endpoint 1.

In order that $1 - \frac{1}{5}\sqrt{15}$ is the midpoint of J, we take the left-hand endpoint to be

$$1 - \tfrac{1}{5}\sqrt{15} - \left(1 - \left(1 - \tfrac{1}{5}\sqrt{15}\right)\right) = 1 - \tfrac{2}{5}\sqrt{15}.$$

Thus an interval of attraction for $1 - \frac{1}{5}\sqrt{15}$ is

$$J = \left(1 - \tfrac{2}{5}\sqrt{15}, 1\right).$$

(d) (i) $x_0 = 0.5$ lies in the interval of attraction I obtained in the solution to part (c)(i) (and also in the interval J obtained in part (c)(ii)), so

$$x_n \to 1 - \tfrac{1}{5}\sqrt{15} \text{ as } n \to \infty.$$

(ii) As $1 - \frac{2}{5}\sqrt{15} \simeq -0.549$, the initial term $x_0 = -0.5$ lies in the interval of attraction J obtained in the solution to part (c)(ii), so

$$x_n \to 1 - \tfrac{1}{5}\sqrt{15} \text{ as } n \to \infty.$$

(iii) $x_0 = 1$ lies in the interval of attraction I obtained in the solution to part (c)(i), so

$$x_n \to 1 - \tfrac{1}{5}\sqrt{15} \text{ as } n \to \infty.$$

(iv) You can check using graphical iteration on your sketch for part (a) that if $x_0 = 2$, then

$$x_n \to \infty \text{ as } n \to \infty.$$

(v) Graphical iteration on your sketch for part (a), starting with $x_0 = -1$, should suggest the sequence converges. As $x_1 = f(x_0) = 0.7$ lies in the interval of attraction I (and also in J), we have

$$x_n \to 1 - \tfrac{1}{5}\sqrt{15} \text{ as } n \to \infty.$$

(vi) As $1 + \frac{1}{5}\sqrt{15}$ is a fixed point of f, $x_n = 1 + \frac{1}{5}\sqrt{15}$ for all n, so

$$x_n \to 1 + \tfrac{1}{5}\sqrt{15} \text{ as } n \to \infty.$$

Solution 2.4

(a) As $f'(x) = x + 1$, we have

$$f'(-\sqrt{2}) = -\sqrt{2} + 1 \simeq -0.414 > -1,$$

so $-\sqrt{2}$ is an attracting fixed point.

(b) As $f'(x) = x + 1$, the condition $|f'(x)| < 1$ can be written as the two inequalities

$$-1 < x + 1 < 1; \quad \text{that is,} \quad -2 < x < 0.$$

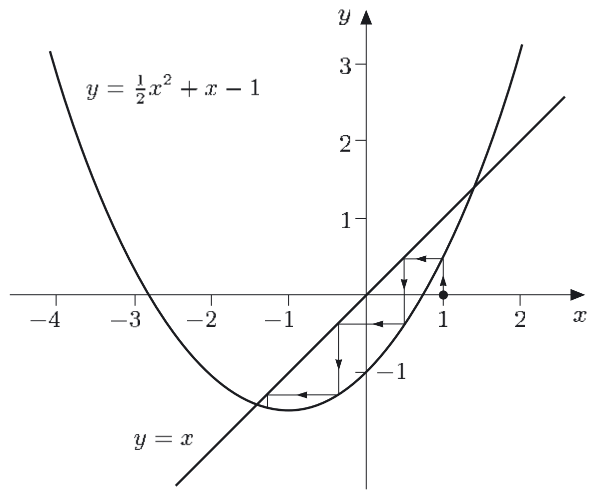
The attracting fixed point $-\sqrt{2} \simeq -1.414$ is nearer to -2 than 0, so we choose I to have left-hand endpoint -2. In order that $-\sqrt{2}$ is the midpoint of I, we take the right-hand endpoint to be

$$-\sqrt{2} + \left(-\sqrt{2} - (-2)\right) = 2 - 2\sqrt{2}.$$

Thus an interval of attraction for $-\sqrt{2}$ is

$$I = (-2, 2 - 2\sqrt{2}).$$

(c) It helps to sketch the graphs of $y = f(x)$ and $y = x$ on the same axes and investigate the effect of graphical iteration with each of the initial terms. We illustrate this effect for $x_0 = 1$.



(i) As $2 - 2\sqrt{2} \simeq -0.828$, the initial term $x_0 = -1$ lies in the interval of attraction I found in part (b), so

$$x_n \to -\sqrt{2} \text{ as } n \to \infty.$$

(ii) Graphical iteration with $x_0 = 1$ suggests that x_n converges to the fixed point $-\sqrt{2}$. Computing the first few values of the sequence gives $x_1 = 0.5$, $x_2 = -0.375$ and $x_3 = -1.304\,687\,5$. As x_3 lies within the interval of attraction I, we deduce that

$$x_n \to -\sqrt{2} \text{ as } n \to \infty.$$

(iii) As

$$\begin{aligned} f(\sqrt{2}) &= \tfrac{1}{2}(\sqrt{2})^2 + \sqrt{2} - 1 \\ &= 1 + \sqrt{2} - 1 = \sqrt{2}, \end{aligned}$$

we deduce that $\sqrt{2}$ is a fixed point of f. Thus if $x_0 = \sqrt{2}$, then $x_n = \sqrt{2}$ for all n, so

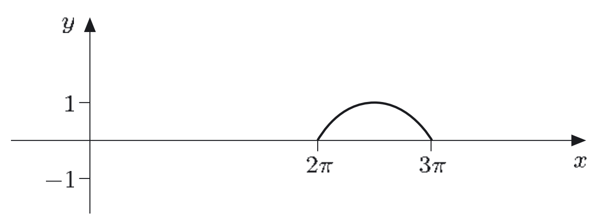
$$x_n \to \sqrt{2} \text{ as } n \to \infty.$$

(iv) If $x_0 = 2$, then graphical iteration gives that

$$x_n \to \infty \text{ as } n \to \infty.$$

Solution 3.1

(a) The domain of f is $A = [2\pi, 3\pi]$ and its codomain is $B = \mathbb{R}$. The graph of $y = f(x)$ is as follows.

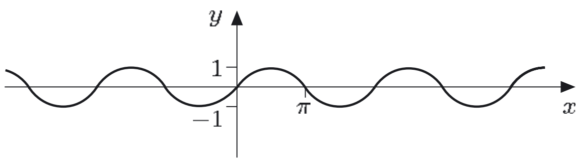


Therefore the image set of f is $f(A) = [0, 1]$.

The domain of g is $C = [0, \infty)$ and its codomain is $D = \mathbb{R}$. Since $[0, 1] \subseteq [0, \infty)$, we have $f(A) \subseteq C$, so the composite function $g \circ f$ exists and is given by

$$\begin{aligned} g \circ f :\ & [2\pi, 3\pi] \longrightarrow \mathbb{R} \\ & x \longmapsto g(f(x)) = \sqrt{\sin x}. \end{aligned}$$

(b) The domain of f is $A = \mathbb{R}$ and its codomain is $B = \mathbb{R}$. The graph of $y = f(x)$ is as follows.



Therefore the image set of f is $f(A) = [-1, 1]$.

The domain of g is $C = [-3, \infty)$ and its codomain is $D = \mathbb{R}$. Since $[-1, 1] \subseteq [-3, \infty)$, we have $f(A) \subseteq C$, so the composite function $g \circ f$ exists and is given by

$$\begin{aligned} g \circ f :\ & \mathbb{R} \longrightarrow \mathbb{R} \\ & x \longmapsto g(f(x)) = \sqrt{\sin x + 3}. \end{aligned}$$

Solution 3.2

(a) We first check that $f(a) = b$ and then that $f(b) = a$:

$$f(\tfrac{3}{7}) = \tfrac{7}{2} \times \tfrac{3}{7} \times \tfrac{4}{7} = \tfrac{6}{7},$$
$$f(\tfrac{6}{7}) = \tfrac{7}{2} \times \tfrac{6}{7} \times \tfrac{1}{7} = \tfrac{3}{7}.$$

Thus $a = \frac{3}{7}$, $b = \frac{6}{7}$ form a 2-cycle of $f(x) = \frac{7}{2}x(1 - x)$.

For $f(x) = \frac{7}{2}x(1 - x) = \frac{7}{2}x - \frac{7}{2}x^2$, we have $f'(x) = -7x + \frac{7}{2}$, so

$$f'(a)f'(b) = \left(-3 + \tfrac{7}{2}\right)\left(-6 + \tfrac{7}{2}\right) = -\tfrac{5}{4} < -1.$$

Thus this 2-cycle is repelling.

(b) In this case

$$\begin{aligned} f\left(-\tfrac{2}{3}\right) &= \left(-\tfrac{2}{3}\right)^2 - \tfrac{7}{9} \\ &= \tfrac{4}{9} - \tfrac{7}{9} \\ &= -\tfrac{1}{3}, \end{aligned}$$

$$\begin{aligned} f\left(-\tfrac{1}{3}\right) &= \left(-\tfrac{1}{3}\right)^2 - \tfrac{7}{9} \\ &= \tfrac{1}{9} - \tfrac{7}{9} \\ &= -\tfrac{2}{3}. \end{aligned}$$

Thus

$$a = -\tfrac{2}{3}, \quad b = -\tfrac{1}{3}$$

form a 2-cycle of $f(x) = x^2 - \frac{7}{9}$.

For $f(x) = x^2 - \frac{7}{9}$, we have $f'(x) = 2x$, so

$$\begin{aligned} f'(a)f'(b) &= (2a)(2b) \\ &= 4(-\tfrac{2}{3})(-\tfrac{1}{3}) \\ &= \tfrac{8}{9} < 1. \end{aligned}$$

Thus this 2-cycle is attracting.

Solution 5.1

(a) There are $5! = 120$ such permutations.

(b) There are

$${}^5P_3 = 5 \times 4 \times 3 = 60$$

such permutations.

(c) There are

$${}^7C_5 = \frac{7!}{2!\,5!} = \frac{7 \times 6 \times 5 \times 4 \times 3}{5 \times 4 \times 3 \times 2 \times 1} = 21$$

such combinations.

Solution 5.2

(a) There are $12! = 479\,001\,600$ such arrangements.

(b) There are

$${}^{12}P_6 = 12 \times 11 \times 10 \times 9 \times 8 \times 7 = 665\,280$$

such arrangements.

(c) The number is the same as that of the six letter arrangements (to follow the initial C) which can be made using letters from the list A, B, D, E, F, G, H, I, J, K, L at most once each, which is

$${}^{11}P_6 = 11 \times 10 \times 9 \times 8 \times 7 \times 6 = 332\,640.$$

(d) The number of ways in which six letters can be selected from the twelve is

$${}^{12}C_6 = \frac{12!}{6!\,6!} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 924.$$

Solution 5.3

(a) The first four terms are:

$$\begin{aligned} & 1 + {}^7C_1(3x) + {}^7C_2(3x)^2 + {}^7C_3(3x)^3 \\ &= 1 + 7(3x) + 21(3x)^2 + 35(3x)^3 \\ &= 1 + 21x + 189x^2 + 945x^3. \end{aligned}$$

(b) The first four terms are:

$$\begin{aligned} & 3^6 + {}^6C_1 \times 3^5(-x^2) + {}^6C_2 \times 3^4(-x^2)^2 \\ & \quad + {}^6C_3 \times 3^3(-x^2)^3 \\ &= 3^6 + 6 \times 3^5(-x^2) + 15 \times 3^4(-x^2)^2 \\ & \quad + 20 \times 3^3(-x^2)^3 \\ &= 729 - 1458x^2 + 1215x^4 - 540x^6. \end{aligned}$$

Solution 5.4

(a) The coefficient of x^8 in the expansion of $(1+x)^{10}$ is

$$\begin{aligned}{}^{10}C_8 &= \frac{10!}{2!\,8!}\\ &= \frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3}{8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}\\ &= 45.\end{aligned}$$

(Alternatively,

$${}^{10}C_8 = \frac{10!}{2!\,8!} = \frac{10\times 9}{2\times 1} = 45,$$

corresponding to the fact that ${}^{10}C_8 = {}^{10}C_2$.)

(b) The x^8 term in the expansion of $(x^2-2)^{10}$ is

$$\begin{aligned}{}^{10}C_4 \times (x^2)^4 \times (-2)^6 &= \frac{10!}{6!\,4!} \times x^8 \times 2^6\\ &= \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} \times x^8 \times 2^6\\ &= 13\,440x^8,\end{aligned}$$

so the required coefficient is 13 440.

(c) The terms in this expansion are of the form

$${}^{11}C_k \times 2^{11-k} \times (x^3)^k,$$

for $k = 0, 1, \ldots, 11$, none of which involves x^8. So the coefficient of x^8 in this expansion is 0.

Solution 5.5

(a) The coefficient of p^7q^3 in the expansion of $(p+q)^{10}$ is

$${}^{10}C_3 = \frac{10!}{7!\,3!} = \frac{10\times 9\times 8}{3\times 2\times 1} = 120.$$

(b) The x^5y^7 term in the expansion of $(\frac{1}{4}x - 2y)^{12}$ is

$$\begin{aligned}&{}^{12}C_7 \times \left(\tfrac{1}{4}x\right)^5 \times (-2y)^7\\ &= \frac{12!}{5!\,7!}\left(\tfrac{1}{4}\right)^5(-2)^7x^5y^7\\ &= \frac{12\times 11\times 10\times 9\times 8\times 7\times 6}{7\times 6\times 5\times 4\times 3\times 2\times 1}\left(\tfrac{1}{4}\right)^5(-2)^7x^5y^7\\ &= -99x^5y^7,\end{aligned}$$

so the required coefficient is -99.

(c) The constant term in the expansion of $\left(x + \dfrac{1}{x^2}\right)^9$ arises from the term which involves the product of x to the power 6 and $\dfrac{1}{x^2}$ to the power 3, and so is given by

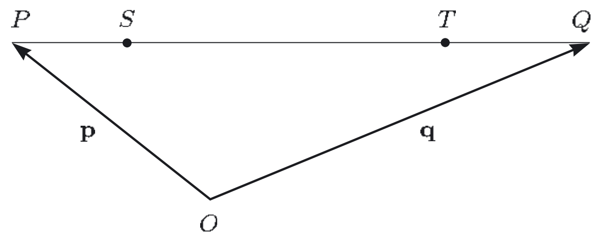
$${}^{9}C_3 = \frac{9!}{6!\,3!} = \frac{9\times 8\times 7}{3\times 2\times 1} = 84.$$

(d) For no value of $k = 0, 1, \ldots, 8$ does the term $x^{8-k}\left(\dfrac{1}{x^2}\right)^k$ become x^0, that is, a constant term. So the constant term in the expansion is 0.

Solutions for Chapter B2

Solution 1.1

(a)



First let $\mathbf{s}$ be the position vector of S. Then

$$\mathbf{s} = \overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{PS} = \mathbf{p} + \tfrac{1}{5}\overrightarrow{PQ}.$$

But $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = -\mathbf{p} + \mathbf{q} = \mathbf{q} - \mathbf{p}$, so

$$\mathbf{s} = \mathbf{p} + \tfrac{1}{5}(\mathbf{q} - \mathbf{p}) = \tfrac{4}{5}\mathbf{p} + \tfrac{1}{5}\mathbf{q}.$$

Next let $\mathbf{t}$ be the position vector of T. Then

$$\mathbf{t} = \overrightarrow{OT} = \overrightarrow{OP} + \overrightarrow{PT} = \mathbf{p} + \tfrac{3}{4}\overrightarrow{PQ},$$

so

$$\mathbf{t} = \mathbf{p} + \tfrac{3}{4}(\mathbf{q} - \mathbf{p}) = \tfrac{1}{4}\mathbf{p} + \tfrac{3}{4}\mathbf{q}.$$

(b) In the particular case where $P = (3, -1)$ and $Q = (-7, 11)$, we have

$$\mathbf{s} = \tfrac{4}{5}\begin{pmatrix}3\\-1\end{pmatrix} + \tfrac{1}{5}\begin{pmatrix}-7\\11\end{pmatrix} = \begin{pmatrix}1\\\frac{7}{5}\end{pmatrix}.$$

and

$$\mathbf{t} = \tfrac{1}{4}\begin{pmatrix}3\\-1\end{pmatrix} + \tfrac{3}{4}\begin{pmatrix}-7\\11\end{pmatrix} = \begin{pmatrix}-\frac{9}{2}\\8\end{pmatrix}.$$

Solution 1.2

The rotation is represented by the matrix

$$\begin{aligned}\mathbf{R}_{2\pi/3} &= \begin{pmatrix}\cos(2\pi/3) & -\sin(2\pi/3)\\ \sin(2\pi/3) & \cos(2\pi/3)\end{pmatrix}\\ &= \begin{pmatrix}-\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix}.\end{aligned}$$

The image of the triangle therefore has vertices at

$$\begin{pmatrix}-\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}-1\\\sqrt{3}\end{pmatrix};$$

$$\begin{pmatrix}-\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\sqrt{3}\\-\frac{1}{2}\end{pmatrix};$$

$$\begin{pmatrix}-\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-\frac{1}{2} - \sqrt{3}\\\frac{1}{2}\sqrt{3} - 1\end{pmatrix}.$$

So, the image is a triangle with vertices at $(-1, \sqrt{3})$, $(-\frac{1}{2}\sqrt{3}, -\frac{1}{2})$ and $(-\frac{1}{2} - \sqrt{3}, \frac{1}{2}\sqrt{3} - 1)$.

Solution 1.3

The reflection is represented by the matrix

$$\mathbf{Q}_{\pi/8} = \begin{pmatrix} \cos(\pi/4) & \sin(\pi/4) \\ \sin(\pi/4) & -\cos(\pi/4) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{pmatrix}.$$

The image of the square therefore has vertices at

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix};$$

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{pmatrix}\begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix};$$

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{pmatrix}\begin{pmatrix} -\sqrt{2} \\ -\sqrt{2} \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix};$$

$$\begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{pmatrix}\begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

So, the image is a square with vertices at $(2,0)$, $(0,-2)$, $(-2,0)$ and $(0,2)$.

Solution 1.4

(a) This matrix has the form

$$\mathbf{Q}_{\theta} = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

where

$$\cos(2\theta) = -\tfrac{1}{2}\sqrt{2} \quad \text{and} \quad \sin(2\theta) = -\tfrac{1}{2}\sqrt{2}.$$

These equations are satisfied by $2\theta = -3\pi/4$, so $\theta = -3\pi/8$.

(b) This matrix has the form

$$\mathbf{R}_{\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where

$$\cos\theta = -\tfrac{1}{2}\sqrt{2} \quad \text{and} \quad \sin\theta = -\tfrac{1}{2}\sqrt{2}.$$

These equations are satisfied by $\theta = -3\pi/4$.

Solution 2.1

(a) f is a linear transformation represented by the matrix

$$\begin{pmatrix} 0 & 0 \\ -3 & 1 \end{pmatrix}.$$

(b) f is not a linear transformation because it maps $(0,0)$ to $(-1,0)$. (Any linear transformation maps the origin to itself.)

(c) f is a linear transformation represented by the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(d) f is not a linear transformation because it maps $(0,0)$ to $(-1,1)$.

(e) f is a linear transformation represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(f) f is not a linear transformation because it maps $(0,0)$ to $(1,-1)$.

Solution 2.2

(a) $\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} \cos(-\pi/3) & \sin(-\pi/3) \\ \sin(-\pi/3) & -\cos(-\pi/3) \end{pmatrix}$

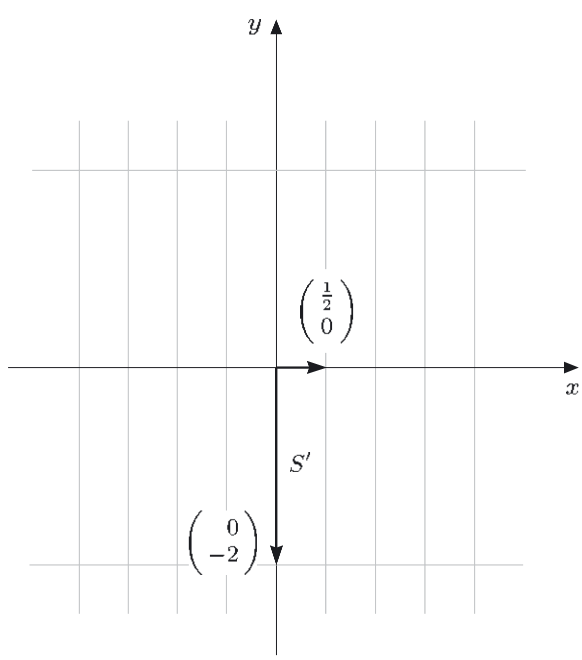
$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} -1 & 3 \\ -2 & 0 \end{pmatrix}$

Solution 2.3

(a) Here f maps $(1,0)$ to the point with position vector $\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$, and it maps $(0,1)$ to the point with position vector $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$, so the unit grid is mapped to the grid shown below.



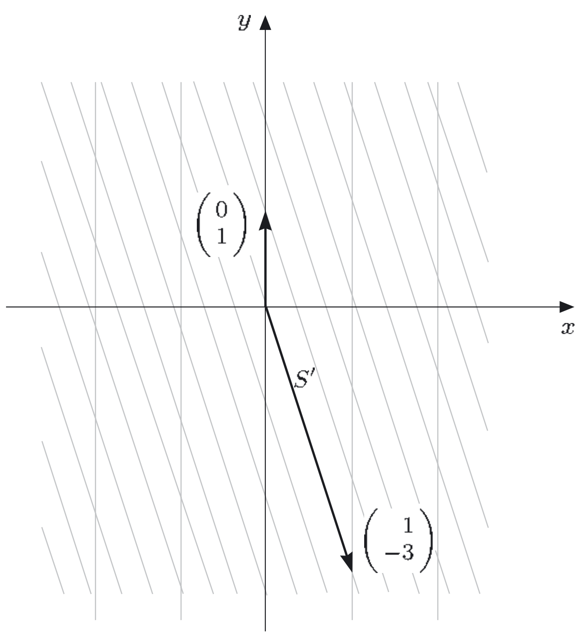
The grid is not skewed, but the negative 2 means that the grid is reflected in the x-axis.

Distances parallel to the x-axis are scaled by the factor $\frac{1}{2}$, and distances parallel to the y-axis are scaled by the factor $|-2| = 2$.

Since the unit square S is mapped to the rectangle S' with vertices at $(0,0)$, $(\frac{1}{2},0)$, $(\frac{1}{2},-2)$, $(0,-2)$, areas are scaled by the factor $\frac{1}{2} \times 2 = 1$, so that areas stay the same. (Alternatively, as $\det \mathbf{A} = -1$, areas are scaled by $|-1| = 1$.)

The reflection in the x-axis means that the transformation reverses orientation. (Alternatively, as $\det \mathbf{A} = -1$, so is negative, the transformation reverses orientation.)

(b) Here f maps $(1,0)$ to the point with position vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$, and it leaves $(0,1)$ unchanged, so the unit grid is mapped to the grid below.



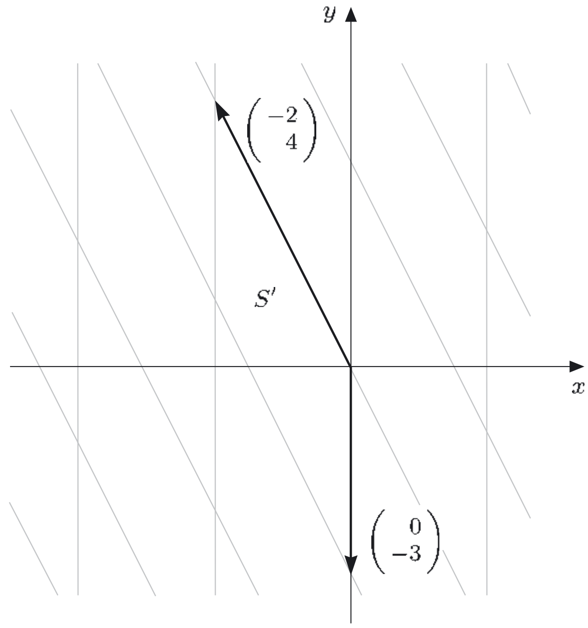
This transformation is a shear parallel to the y-axis with factor -3. Each point moves parallel to the y-axis through a distance proportional (by a factor $|-3| = 3$) to its horizontal distance from the y-axis. As the factor -3 is negative, points to the right of the y-axis move down and those to the left of the y-axis move up.

Although the transformation changes the shape of the unit square into a parallelogram, the area of this parallelogram is the same as the area of the unit square. Thus areas stay the same (as is confirmed by $\det \mathbf{A} = 1$).

Also, under the shear, the ordering of the vertices of the unit square remains unchanged, so the shear preserves orientation. (Alternatively, as $\det \mathbf{A} = 1$, so is positive, the transformation preserves orientation.)

Solution 2.4

Here f maps $(1,0)$ to the point with position vector $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$, and it maps $(0,1)$ to the point with position vector $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$, so the unit grid is mapped to the grid shown below.



As $\det \mathbf{A} = 6$, areas are scaled by 6.

As $\det \mathbf{A}$ is positive, the transformation preserves orientation.

Solution 2.5

(a) This matrix represents a scaling with factors -1 and -3; that is, the plane is scaled by the factor -1 parallel to the x-axis, and by the factor -3 parallel to the y-axis.

We have

$$\det \mathbf{A} = (-1) \times (-3) - 0 \times 0 = 3.$$

It follows that areas are scaled by the factor 3 and, as $\det \mathbf{A}$ is positive, orientation is preserved.

(b) This matrix has the form

$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

with $\cos(2\theta) = 1$ and $\sin(2\theta) = 0$. Thus the value of 2θ is 0. So f is the reflection in the line through the origin that makes an angle 0 with the positive x-axis, namely the x-axis itself. (It is perhaps much quicker with this simple matrix to work out the effect of f on the unit grid and observe that it is reflection in the x-axis.)

We have

$$\det \mathbf{A} = 1 \times (-1) - 0 \times 0 = -1.$$

It follows that areas are scaled by the factor $|-1| = 1$, which means that areas are preserved, and, as $\det \mathbf{A}$ is negative, orientation is reversed.

(c) This matrix has the form

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$$

with $a = 4$. So f is a y-shear with factor 4.

We have

$$\det \mathbf{A} = 1 \times 1 - 0 \times 4 = 1.$$

It follows that areas and orientation are both preserved.

(d) The first column of this matrix is $-\frac{3}{2}$ times the second column, so this matrix represents a flattening. Indeed, the image of an arbitrary point (x, y) has position vector

$$x\begin{pmatrix} -6 \\ 9 \end{pmatrix} + y\begin{pmatrix} 4 \\ -6 \end{pmatrix} = (3x - 2y)\begin{pmatrix} -2 \\ 3 \end{pmatrix},$$

so f is a flattening onto the line $3x + 2y = 0$ (through the origin and the point $(-2, 3)$).

We have

$$\det \mathbf{A} = (-6) \times (-6) - 4 \times 9 = 0.$$

This is consistent with the fact that a flattening scales areas to 0. Orientation is destroyed.

Solution 2.6

Since f sends $(1, 0)$ to $(4, 2)$ and $(0, 1)$ to $(-2, -7)$, it is represented by the matrix

$$\begin{pmatrix} 4 & -2 \\ 2 & -7 \end{pmatrix}.$$

This has determinant $4 \times (-7) - (-2) \times 2 = -24$, so f scales areas by a factor of 24. The triangle T is the image of the right-angled triangle with vertices at $(0, 0)$, $(1, 0)$, $(0, 1)$. Since this right-angled triangle has area $\frac{1}{2}$, it follows that T has area $24 \times \frac{1}{2} = 12$.

Solution 2.7

(a) Suppose (r, s) and (u, v) are points such that $f(r, s) = f(u, v)$. Then we have

$$\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix};$$

that is,

$$\begin{pmatrix} r + 3s \\ 2r + 7s \end{pmatrix} = \begin{pmatrix} u + 3v \\ 2u + 7v \end{pmatrix}.$$

Equating components, we have $r + 3s = u + 3v$ and $2r + 7s = 2u + 7v$.

Subtracting two times the first equation from the second equation gives $s = v$, from which we obtain $r = u$. It follows that $(r, s) = (u, v)$. Hence f is one-one.

(b) Let (u, v) be an arbitrary point in the codomain $\mathbb{R}^2$. For (u, v) to be the image of a point (x, y) in the domain, we require $f(x, y) = (u, v)$; that is,

$$\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

or, equivalently,

$$\begin{pmatrix} x + 3y \\ 2x + 7y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Solving for (x, y) in terms of u and v, we obtain $x = 7u - 3v$ and $y = v - 2u$. Thus one point that maps to (u, v) is $(7u - 3v, v - 2u)$. Since (u, v) is an arbitrary point in the codomain $\mathbb{R}^2$, we conclude that $f(\mathbb{R}^2) = \mathbb{R}^2$. Hence f is onto.

Solution 3.1

(a)
$$\begin{aligned}\mathbf{Q}_{\pi/8} &= \begin{pmatrix} \cos(\pi/4) & \sin(\pi/4) \\ \sin(\pi/4) & -\cos(\pi/4) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{pmatrix};\end{aligned}$$

$$\begin{aligned}\mathbf{R}_0 &= \begin{pmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}.\end{aligned}$$

(b) We have

$$\begin{aligned}&\mathbf{Q}_{\pi/8}\mathbf{Q}_{\pi/8} \\ &= \begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}(\sqrt{2})^2 + \frac{1}{4}(\sqrt{2})^2 & \frac{1}{4}(\sqrt{2})^2 - \frac{1}{4}(\sqrt{2})^2 \\ \frac{1}{4}(\sqrt{2})^2 - \frac{1}{4}(\sqrt{2})^2 & \frac{1}{4}(\sqrt{2})^2 + \frac{1}{4}(\sqrt{2})^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \mathbf{R}_0.\end{aligned}$$

Solution 3.2

(a) The composite $g \circ f$ is the linear transformation represented by the matrix

$$\mathbf{BA} = \begin{pmatrix} -1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 29 & 8 \end{pmatrix}.$$

(b) The composite $f \circ g$ is the linear transformation represented by the matrix

$$\mathbf{AB} = \begin{pmatrix} 5 & 2 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 16 \\ -3 & 6 \end{pmatrix}.$$

Solution 3.3

(a) f is a scaling with factors 4 and 3, and g is an x-shear with factor -2.

(b) The composite $g \circ f$ is the linear transformation represented by the matrix

$$\begin{aligned}\mathbf{BA} &= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -6 \\ 0 & 3 \end{pmatrix} \\ &= \mathbf{C},\end{aligned}$$

so $g \circ f = h$.

Solution 3.4

(a) In this case

$$\det \mathbf{A} = 4 \times 5 - 3 \times 8 = -4.$$

Since this determinant is non-zero, it follows by the result on page 42 of the chapter that f is one-one and onto.

(b) The inverse f^{-1} is the linear transformation represented by the matrix

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{pmatrix} 5 & -3 \\ -8 & 4 \end{pmatrix} = \begin{pmatrix} -\frac{5}{4} & \frac{3}{4} \\ 2 & -1 \end{pmatrix}.$$

(c) The required point is $f^{-1}(1,3)$ with position vector

$$\mathbf{A}^{-1}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{5}{4} & \frac{3}{4} \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Solution 3.5

(a) Here

$$\det \mathbf{A} = 1 \times 6 - (-2) \times (-3) = 0.$$

Since this determinant is zero, it follows that $\mathbf{A}$ is not invertible.

(b) In this case

$$\det \mathbf{A} = 0 \times 6 - 1 \times 3 = -3.$$

Since this determinant is non-zero, it follows that $\mathbf{A}$ is invertible. Moreover,

$$\mathbf{A}^{-1} = \frac{1}{-3}\begin{pmatrix} 6 & -1 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} -2 & \frac{1}{3} \\ 1 & 0 \end{pmatrix}.$$

(c) Here

$$\det \mathbf{A} = 2 \times (-2) - 1 \times (-6) = 2.$$

Since this determinant is non-zero, it follows that $\mathbf{A}$ is invertible. Moreover,

$$\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} -2 & -1 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -\frac{1}{2} \\ 3 & 1 \end{pmatrix}.$$

Solution 3.6

(a) The matrix that represents f is

$$\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}.$$

The matrix that represents g is

$$\mathbf{B} = \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}.$$

(b) First observe that

$$\det \mathbf{A} = (-1) \times (-3) - 2 \times 2 = -1,$$

so f has an inverse transformation f^{-1}. The required linear transformation is therefore f^{-1}, which is represented by the matrix

$$\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} -3 & -2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}.$$

(c) We know that f^{-1} sends $(-1,2)$ and $(2,-3)$ to the points $(1,0)$ and $(0,1)$, respectively. We also know that g sends $(1,0)$ and $(0,1)$ to the points $(3,4)$ and $(1,-1)$, respectively. The required linear transformation is therefore the composite $g \circ f^{-1}$. This is represented by the matrix

$$\begin{aligned}\mathbf{BA}^{-1} &= \begin{pmatrix} 3 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 11 & 7 \\ 10 & 7 \end{pmatrix}.\end{aligned}$$

(As a check, notice that

$$\begin{pmatrix} 11 & 7 \\ 10 & 7 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

and

$$\begin{pmatrix} 11 & 7 \\ 10 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

as required.)

Solution 3.7

First observe that $\det \mathbf{A} = 2 \times 2 - 1 \times 1 = 3$, so f has an inverse transformation f^{-1} represented by the matrix

$$\mathbf{A}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

If P is an arbitrary point (x,y) on the image $f(\mathscr{C})$, then P must be the image under f of the point $f^{-1}(P)$ on $\mathscr{C}$ with position vector

$$\mathbf{A}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{3}x - \frac{1}{3}y \\ -\frac{1}{3}x + \frac{2}{3}y \end{pmatrix}.$$

Since these are the coordinates of a point on $\mathscr{C}$, it follows that

$$(\tfrac{2}{3}x - \tfrac{1}{3}y)^2 + (-\tfrac{1}{3}x + \tfrac{2}{3}y)^2 = 1,$$

or, equivalently,

$$\tfrac{4}{9}x^2 - \tfrac{4}{9}xy + \tfrac{1}{9}y^2 + \tfrac{1}{9}x^2 - \tfrac{4}{9}xy + \tfrac{4}{9}y^2 = 1.$$

That is,

$$\tfrac{5}{9}x^2 - \tfrac{8}{9}xy + \tfrac{5}{9}y^2 = 1,$$

or, equivalently,

$$5x^2 - 8xy + 5y^2 = 9.$$

This is therefore the equation of $f(\mathscr{C})$.

The area enclosed by $\mathscr{C}$ is π and f scales areas by the factor $|\det \mathbf{A}| = 3$, so the area enclosed by $f(\mathscr{C})$ is 3π.

Solution 4.1

(a) The required affine transformation has the rule $f\colon \mathbf{x} \longmapsto \mathbf{A}\mathbf{x} + \mathbf{a}$, where

$$\mathbf{A} = \begin{pmatrix} 7-(-1) & 3-(-1) \\ 5-(-3) & 0-(-3) \end{pmatrix} = \begin{pmatrix} 8 & 4 \\ 8 & 3 \end{pmatrix}$$

and

$$\mathbf{a} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}.$$

(b) It follows that the affine transformation scales areas by

$$|\det \mathbf{A}| = |8 \times 3 - 4 \times 8| = 8.$$

Since the triangle with vertices at $(0,0)$, $(1,0)$, $(0,1)$ has area $\frac{1}{2}$, the area of T must be $8 \times \frac{1}{2} = 4$.

Solution 4.2

The rotation is given by the composite $t_{4,2} \circ (r_{\pi/3} \circ t_{-4,-2})$. Under this composite transformation, an arbitrary point $\mathbf{x}$ is mapped to

$$\begin{aligned}
&\mathbf{R}_{\pi/3}\left(\mathbf{x} + \begin{pmatrix} -4 \\ -2 \end{pmatrix}\right) + \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 4 \\ 2 \end{pmatrix}\right) + \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}\mathbf{x} - \begin{pmatrix} 2-\sqrt{3} \\ 2\sqrt{3}+1 \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2+\sqrt{3} \\ 1-2\sqrt{3} \end{pmatrix}.
\end{aligned}$$

So the given rotation about the point $(4,2)$ can be expressed as the affine transformation $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(\mathbf{x}) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}\mathbf{x} + \begin{pmatrix} 2+\sqrt{3} \\ 1-2\sqrt{3} \end{pmatrix}.$$

Solution 4.3

(a) Since the y-axis makes an angle $\pi/2$ with the positive x-axis, it follows that the reflection in the y-axis is $q_{\pi/2}$, so the required matrix is

$$\mathbf{Q}_{\pi/2} = \begin{pmatrix} \cos(\pi) & \sin(\pi) \\ \sin(\pi) & -\cos(\pi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Alternatively, notice that the images of $(1,0)$ and $(0,1)$ under the reflection are $(-1,0)$ and $(0,1)$, respectively. The position vectors of these images form the columns of the matrix.)

(b) The reflection is given by the composite $t_{3,0} \circ (q_{\pi/2} \circ t_{-3,0})$. Under this composite transformation, an arbitrary point $\mathbf{x}$ is mapped to

$$\begin{aligned}
&\mathbf{Q}_{\pi/2}\left(\mathbf{x} + \begin{pmatrix} -3 \\ 0 \end{pmatrix}\right) + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 3 \\ 0 \end{pmatrix}\right) + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 6 \\ 0 \end{pmatrix}.
\end{aligned}$$

So the reflection in the line $x = 3$ can be expressed as the affine transformation $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$f(\mathbf{x}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 6 \\ 0 \end{pmatrix}.$$

(An alternative way of doing this is to note that the images of $(0,0)$, $(1,0)$ and $(0,1)$ under the reflection are respectively $(6,0)$, $(5,0)$ and $(6,1)$. As we know that the reflection is an affine transformation, we can then write down its rule as

$$\begin{aligned}
f(\mathbf{x}) &= \begin{pmatrix} 5-6 & 6-6 \\ 0-0 & 1-0 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 6 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 6 \\ 0 \end{pmatrix}.)
\end{aligned}$$

Solutions for Chapter B3

Solution 1.1

(a) Every point is fixed by the rotation $r_{-2\pi}$, as -2π is an integer multiple of 2π. Every line through the origin is an invariant line.

(b) The reflection $q_{\pi/3}$ has a line of fixed points, the line through the origin making an angle $\pi/3$ with the positive x-axis (which has equation $y = \sqrt{3}x$). It has two invariant lines through the origin, the line $y = \sqrt{3}x$ (which is a line of fixed points) and the line through the origin perpendicular to this, making an angle $5\pi/6$ with the positive x-axis (which has equation $y = -\frac{1}{\sqrt{3}}x$).

(c) The scaling with factors -4 and 1 fixes all points on the y-axis. It has two invariant lines through the origin, the x-axis and the y-axis (which is a line of fixed points).

(d) The y-shear with factor -1 has a line of fixed points, the y-axis. It has only one invariant line through the origin, the y-axis (which is a line of fixed points).

Solution 1.2

(a) $$\mathbf{Q}_{-\pi/4} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

To show that every point on the line $y = -x$ is a fixed point, consider an arbitrary point on the line, say $(c, -c)$. The image of the point $(c, -c)$ is given by

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} c \\ -c \end{pmatrix} = \begin{pmatrix} c \\ -c \end{pmatrix},$$

so the point $(c, -c)$ is a fixed point. Since $(c, -c)$ is an arbitrary point on the line $y = -x$, we have shown that every point on this line is a fixed point.

(b) To show that the x-axis is not an invariant line, we must show that the image of some point on the x-axis is not on the x-axis. Consider the point $(1, 0)$. The image of the point $(1, 0)$ is given by

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The point $(0, -1)$ does not lie on the x-axis, so the x-axis is not an invariant line of this reflection.

(In fact, no point on the x-axis other than the origin has its image on the x-axis.)

Solution 2.1

(a) The eigenvector equation with eigenvalue 2,

$$\begin{pmatrix} -\frac{4}{3} & \frac{14}{3} \\ \frac{10}{3} & -\frac{8}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix},$$

gives the two equations

$$-\tfrac{4}{3}x + \tfrac{14}{3}y = 2x,$$
$$\tfrac{10}{3}x - \tfrac{8}{3}y = 2y.$$

These equations both reduce to the equation $\frac{10}{3}x - \frac{14}{3}y = 0$, so the eigenline corresponding to the eigenvalue 2 has equation $y = \frac{5}{7}x$.

Any vector of the form $\begin{pmatrix} 7c \\ 5c \end{pmatrix}$ $(c \neq 0)$ is an eigenvector for this eigenline, so two such eigenvectors are $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} -14 \\ -10 \end{pmatrix}$.

(b) The image of a typical point $(c, -c)$ on the line with equation $y = -x$ is given by

$$\begin{pmatrix} -\frac{4}{3} & \frac{14}{3} \\ \frac{10}{3} & -\frac{8}{3} \end{pmatrix}\begin{pmatrix} c \\ -c \end{pmatrix} = \begin{pmatrix} -6c \\ 6c \end{pmatrix} = -6\begin{pmatrix} c \\ -c \end{pmatrix}.$$

The point $(c, -c)$ has been scaled by -6, so $y = -x$ is indeed an eigenline, with eigenvalue -6.

Solution 2.2

(a) $$\mathbf{A}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{36}{5} \\ -\frac{6}{5} & \frac{8}{5} \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ 4 \end{pmatrix} = 4\begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

so $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to eigenvalue 4. The corresponding eigenline, going through the point $(-2, 1)$ and the origin, has equation $y = -\frac{1}{2}x$.

(b) One way of proceeding is to find the eigenvalues of $\mathbf{A}$ by solving the characteristic equation of $\mathbf{A}$ and seeing whether 2 is one of the solutions. The characteristic equation is

$$k^2 - (\tfrac{2}{5} + \tfrac{8}{5})k + \tfrac{16}{25} - \tfrac{216}{25} = 0;$$

that is,

$$k^2 - 2k - 8 = 0.$$

This factorises as $(k - 4)(k + 2) = 0$, so the eigenvalues are 4 and -2. Thus 2 is not an eigenvalue.

An alternative way of proceeding is to substitute $k = 2$ into the eigenvector equation $\mathbf{A}\mathbf{x} = k\mathbf{x}$ and investigate whether the two resulting equations both reduce to the same equation. If they do, this gives the equation of an eigenline and 2 must be an eigenvalue; if they do not, 2 is not an eigenvalue. The eigenvector equation is

$$\begin{pmatrix} \frac{2}{5} & -\frac{36}{5} \\ -\frac{6}{5} & \frac{8}{5} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix},$$

which corresponds to the equations

$$\begin{aligned} \tfrac{2}{5}x - \tfrac{36}{5}y &= 2x, \\ -\tfrac{6}{5}x + \tfrac{8}{5}y &= 2y. \end{aligned}$$

These do not reduce to the same equation, so 2 is not an eigenvalue.

(c) One way of proceeding is to solve the characteristic equation to find the eigenvalues of $\mathbf{A}$, then find the corresponding eigenlines and thereby discover whether one of these has equation $y = \frac{1}{3}x$. However, it is much quicker to check whether a typical point $(3c, c)$ on the line with equation $y = \frac{1}{3}x$ is mapped to a point on this line by the transformation represented by $\mathbf{A}$. We have

$$\begin{aligned} \mathbf{A}\begin{pmatrix} 3c \\ c \end{pmatrix} &= \begin{pmatrix} \frac{2}{5} & -\frac{36}{5} \\ -\frac{6}{5} & \frac{8}{5} \end{pmatrix} \begin{pmatrix} 3c \\ c \end{pmatrix} \\ &= \begin{pmatrix} -6c \\ -2c \end{pmatrix} \\ &= -2\begin{pmatrix} 3c \\ c \end{pmatrix}. \end{aligned}$$

Thus the point $(3c, c)$ is scaled by a factor -2, so the line with equation $y = \frac{1}{3}x$ is an eigenline.

Solution 2.3

(a) The matrix $\begin{pmatrix} 2 & 6 \\ 2 & 3 \end{pmatrix}$ has characteristic equation

$$k^2 - 5k - 6 = 0.$$

This factorises as $(k-6)(k+1) = 0$, so the eigenvalues of this matrix are 6 and -1.

The eigenvector equation with eigenvalue 6,

$$\begin{pmatrix} 2 & 6 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6 \begin{pmatrix} x \\ y \end{pmatrix},$$

gives the two equations

$$\begin{aligned} 2x + 6y &= 6x, \\ 2x + 3y &= 6y. \end{aligned}$$

These equations both reduce to the equation $y = \frac{2}{3}x$, which is the eigenline corresponding to the eigenvalue 6. A corresponding eigenvector is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

The eigenvector equation with eigenvalue -1,

$$\begin{pmatrix} 2 & 6 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix},$$

gives the two equations

$$\begin{aligned} 2x + 6y &= -x, \\ 2x + 3y &= -y. \end{aligned}$$

These equations both reduce to the equation $y = -\frac{1}{2}x$, which is the eigenline corresponding to the eigenvalue -1. A corresponding eigenvector is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

(b) The matrix $\begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix}$ is a triangular matrix, so its eigenvalues are the elements on the main diagonal. Thus this matrix has only one eigenvalue, namely 2.

The eigenvector equation with eigenvalue 2,

$$\begin{pmatrix} 2 & 5 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix},$$

gives the two equations

$$\begin{aligned} 2x + 5y &= 2x, \\ 2y &= 2y. \end{aligned}$$

The first equation reduces to $y = 0$, while the second equation holds for every value of y (and x). Thus the x-axis is the only eigenline for the matrix. A corresponding eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(c) The matrix $\begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}$ is a triangular matrix, so its eigenvalues are the elements on the main diagonal. Thus this matrix has eigenvalues 2 and 3.

The eigenvector equation with eigenvalue 2,

$$\begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix},$$

gives the two equations

$$\begin{aligned} 2x &= 2x, \\ 5x + 3y &= 2y. \end{aligned}$$

The first equation holds for all values of x (and y). The second equation reduces to $y = -5x$, which is the eigenline corresponding to the eigenvalue 2. A corresponding eigenvector is $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$.

The eigenvector equation with eigenvalue 3,

$$\begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix},$$

gives the two equations

$$\begin{aligned} 2x &= 3x, \\ 5x + 3y &= 3y. \end{aligned}$$

These equations both reduce to $x = 0$. Thus the y-axis is the eigenline corresponding to the eigenvalue 3. A corresponding eigenvector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(d) The matrix $\begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}$ has characteristic equation

$$k^2 + 4k - 5 = 0.$$

This factorises as $(k-1)(k+5) = 0$, so the eigenvalues of this matrix are 1 and -5.

The eigenvector equation with eigenvalue 1,

$$\begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix},$$

gives the two equations

$$\begin{aligned} -x + 4y &= x, \\ 2x - 3y &= y. \end{aligned}$$

These equations both reduce to the equation $y = \frac{1}{2}x$, which is the eigenline corresponding to the eigenvalue 1. A corresponding eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

The eigenvector equation with eigenvalue -5,

$$\begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix},$$

gives the two equations

$$\begin{aligned} -x + 4y &= -5x, \\ 2x - 3y &= -5y. \end{aligned}$$

These equations both reduce to the equation $y = -x$, which is the eigenline corresponding to the eigenvalue -5. A corresponding eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(e) The matrix $\begin{pmatrix} 0 & 2 \\ -2 & 1 \end{pmatrix}$ has characteristic equation

$$k^2 - k + 4 = 0.$$

This equation has no real solutions for k, so this matrix has no eigenvalues or eigenlines.

(f) The matrix $\begin{pmatrix} 1 & -1 \\ -\frac{2}{3} & \frac{4}{3} \end{pmatrix}$ has characteristic equation

$$k^2 - \tfrac{7}{3}k + \tfrac{2}{3} = 0.$$

This factorises as $(k - \frac{1}{3})(k-2) = 0$, so the eigenvalues of this matrix are $\frac{1}{3}$ and 2.

The eigenvector equation with eigenvalue $\frac{1}{3}$,

$$\begin{pmatrix} 1 & -1 \\ -\frac{2}{3} & \frac{4}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix},$$

gives the two equations

$$\begin{aligned} x - y &= \tfrac{1}{3}x, \\ -\tfrac{2}{3}x + \tfrac{4}{3}y &= \tfrac{1}{3}y. \end{aligned}$$

These equations both reduce to the equation $y = \frac{2}{3}x$, which is the eigenline corresponding to the eigenvalue $\frac{1}{3}$. A corresponding eigenvector is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

The eigenvector equation with eigenvalue 2,

$$\begin{pmatrix} 1 & -1 \\ -\frac{2}{3} & \frac{4}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix},$$

gives the two equations

$$\begin{aligned} x - y &= 2x, \\ -\tfrac{2}{3}x + \tfrac{4}{3}y &= 2y. \end{aligned}$$

These equations both reduce to the equation $y = -x$, which is the eigenline corresponding to the eigenvalue 2. A corresponding eigenvector is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Solution 3.1

(a) The eigenvalues of $\mathbf{A}$ are the elements on the leading diagonal of $\mathbf{D}$, namely 2 and -3.

The columns of $\mathbf{P}$ give eigenvectors corresponding to the eigenvalues; and from the eigenvectors we can write down the equations of the eigenlines.

The first column of $\mathbf{P}$ gives the eigenvector $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$, which corresponds to the eigenvalue 2, the first element in the leading diagonal of $\mathbf{D}$. The corresponding eigenline has equation $y = \frac{5}{3}x$.

The second column of $\mathbf{P}$ gives the eigenvector $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, which corresponds to the eigenvalue -3, the second element in the leading diagonal of $\mathbf{D}$. The corresponding eigenline has equation $y = 2x$.

(b) Noting that

$$\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix}$$

and

$$\mathbf{D}^3 = \begin{pmatrix} 8 & 0 \\ 0 & -27 \end{pmatrix},$$

we have

$$\begin{aligned}\mathbf{A}^3 &= \mathbf{P}\mathbf{D}^3\mathbf{P}^{-1} \\ &= \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & -27 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 24 & -54 \\ 40 & -108 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ -5 & 3 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 366 & -210 \\ 700 & -404 \end{pmatrix} \\ &= \begin{pmatrix} 183 & -105 \\ 350 & -202 \end{pmatrix}.\end{aligned}$$

Solution 3.2

(a)

$$\begin{pmatrix} 8 & 3 \\ -18 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

$$\begin{pmatrix} 8 & 3 \\ -18 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

(b) In Step 5 we obtain $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix}$, so in Step 6 we obtain

$$\mathbf{P}^{-1} = \frac{1}{-1}\begin{pmatrix} -3 & -1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix}.$$

Multiplying out the resulting matrix product gives

$$\begin{aligned}\mathbf{P}\mathbf{D}\mathbf{P}^{-1} &= \begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 3 \\ -18 & -7 \end{pmatrix} = \mathbf{A},\end{aligned}$$

as required.

(c) In Step 5 we obtain $\mathbf{P} = \begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}$, so in Step 6 we obtain

$$\mathbf{P}^{-1} = \frac{1}{-2}\begin{pmatrix} -4 & -2 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

Multiplying out the resulting matrix product gives

$$\begin{aligned}\mathbf{P}\mathbf{D}\mathbf{P}^{-1} &= \begin{pmatrix} -1 & 2 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 4 \\ -3 & -8 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 8 & 3 \\ -18 & -7 \end{pmatrix} = \mathbf{A},\end{aligned}$$

as required.

(d) Using our answer to part (b) above, with $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix}$, we obtain

$$\begin{aligned}\mathbf{A}^4 &= \mathbf{P}\mathbf{D}^4\mathbf{P}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 16 & 1 \\ -32 & -3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 46 & 15 \\ -90 & -29 \end{pmatrix}.\end{aligned}$$

Solution 3.3

We follow the strategy, using the results in our solutions to the relevant parts of Exercise 2.3.

(a) Step 1. The eigenvalues of this matrix are 6 and -1.

Step 2. Let $\mathbf{D} = \begin{pmatrix} 6 & 0 \\ 0 & -1 \end{pmatrix}$.

Step 3. The eigenlines of this matrix are $y = \frac{2}{3}x$ and $y = -\frac{1}{2}x$.

Step 4. An eigenvector for the eigenvalue 6 is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. An eigenvector for the eigenvalue -1 is $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Step 5. Let $\mathbf{P} = \begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix}$.

Step 6. Thus

$$\mathbf{P}^{-1} = \frac{1}{-7}\begin{pmatrix} -1 & -2 \\ -2 & 3 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}.$$

Finally, we have

$$\begin{pmatrix} 2 & 6 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 6 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{7}\begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}.$$

In this and the remaining parts, different but correct solutions can be obtained. For instance, taking the eigenvalues in a different order, -1 and 6 rather than 6 and -1, gives at Step 2 the matrix $\mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}$.

Taking the same corresponding eigenvectors as in our solution above gives at Step 5 the matrix $\mathbf{P} = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}$ and at Step 6, the corresponding inverse matrix $\mathbf{P}^{-1} = \frac{1}{7}\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}$. This gives

$$\begin{pmatrix} 2 & 6 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 6 \end{pmatrix}\frac{1}{7}\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix}.$$

Other solutions can be obtained by taking different eigenvectors at Step 4 from those we have chosen to create the matrix $\mathbf{P}$, giving different matrices at Step 5 and 6. You can of course check that your answer is correct by multiplying out the product $\mathbf{PDP}^{-1}$.

(b) Step 1. The eigenvalues of this matrix are 2 and 3.

Step 2. Let $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Step 3. The eigenlines of this matrix are $y = -5x$ and $x = 0$.

Step 4. An eigenvector for the eigenvalue 2 is $\begin{pmatrix} 1 \\ -5 \end{pmatrix}$. An eigenvector for the eigenvalue 3 is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Step 5. Let $\mathbf{P} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$.

Step 6. Thus

$$\mathbf{P}^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}.$$

Finally, we have

$$\begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}.$$

(c) Step 1. The eigenvalues of this matrix are 1 and -5.

Step 2. Let $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}$.

Step 3. The eigenlines of this matrix are $y = \frac{1}{2}x$ and $y = -x$.

Step 4. An eigenvector for the eigenvalue 1 is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. An eigenvector for the eigenvalue -5 is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Step 5. Let $\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$.

Step 6. Thus

$$\mathbf{P}^{-1} = \frac{1}{-3}\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}.$$

Finally, we have

$$\begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}.$$

(d) Step 1. The eigenvalues of this matrix are $\frac{1}{3}$ and 2.

Step 2. Let $\mathbf{D} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix}$.

Step 3. The eigenlines of this matrix are $y = \frac{2}{3}x$ and $y = -x$.

Step 4. An eigenvector for the eigenvalue $\frac{1}{3}$ is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. An eigenvector for the eigenvalue 2 is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Step 5. Let $\mathbf{P} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$.

Step 6. Thus

$$\mathbf{P}^{-1} = \frac{1}{-5}\begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}.$$

Finally, we have

$$\begin{pmatrix} 1 & -1 \\ -\frac{2}{3} & \frac{4}{3} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix}.$$

Solution 4.1

First note that

$$\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned}
\mathbf{A}^n &= \mathbf{PD}^n\mathbf{P}^{-1} \\
&= \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 2^n & 0 \\ 0 & (-\frac{1}{2})^n \end{pmatrix}\frac{1}{2}\begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 2^n & (-\frac{1}{2})^n \\ 2(2^n) & 4(-\frac{1}{2})^n \end{pmatrix}\begin{pmatrix} 4 & -1 \\ -2 & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 4(2^n) - 2(-\frac{1}{2})^n & -(2^n) + (-\frac{1}{2})^n \\ 8(2^n) - 8(-\frac{1}{2})^n & -2(2^n) + 4(-\frac{1}{2})^n \end{pmatrix}.
\end{aligned}$$

Solution 4.2

(a) The initial point $(-6, 4)$ lies on the eigenline $y = -\frac{2}{3}x$, with eigenvalue -3. Hence from the table on page 41 of Chapter B3, the iteration sequence moves away from $(0, 0)$, alternating between the halves of the line $y = -\frac{2}{3}x$.

(b) The initial point $(-8,-8)$ is not on either eigenline. We use the iteration properties of generalised scalings.

By iteration property (a): since $-2<0$, the points of the sequence (x_n, y_n) alternate between opposite sides of the eigenline $y=-\frac{2}{3}x$; since $-3<0$, the points alternate between opposite sides of the eigenline $y=-\frac{1}{2}x$.

By iteration property (b): since $\max\{|-2|,|-3|\}=3>1$, the sequence moves away from $(0,0)$.

Since $|-3|>|-2|$, the dominant eigenvalue is -3 and the dominant eigenline is $y=-\frac{2}{3}x$. Hence by iteration property (c) (the Dominant Eigenvalue Property)

$$\frac{y_n}{x_n} \to -\tfrac{2}{3} \text{ as } n\to\infty.$$

Thus the sequence tends in the direction of the line $y=-\frac{2}{3}x$, moving away from the origin. The points alternate between opposite sides of that line and alternate between opposite sides of the line $y=-\frac{1}{2}x$.

Solution 4.3

(a) (i) First note that

$$\mathbf{P}^{-1}=\frac{1}{2}\begin{pmatrix}1&1\\1&3\end{pmatrix},$$

so

$$\begin{aligned}\mathbf{PDP}^{-1}&=\begin{pmatrix}3&-1\\-1&1\end{pmatrix}\begin{pmatrix}2&0\\0&-4\end{pmatrix}\frac{1}{2}\begin{pmatrix}1&1\\1&3\end{pmatrix}\\&=\begin{pmatrix}6&4\\-2&-4\end{pmatrix}\frac{1}{2}\begin{pmatrix}1&1\\1&3\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}10&18\\-6&-14\end{pmatrix}\\&=\begin{pmatrix}5&9\\-3&-7\end{pmatrix}=\mathbf{A},\end{aligned}$$

as required.

(ii) Since $\mathbf{A}=\mathbf{PDP}^{-1}$, we have

$$\begin{aligned}\mathbf{A}^n&=\mathbf{PD}^n\mathbf{P}^{-1}\\&=\frac{1}{2}\begin{pmatrix}3&-1\\-1&1\end{pmatrix}\begin{pmatrix}2^n&0\\0&(-4)^n\end{pmatrix}\begin{pmatrix}1&1\\1&3\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}3(2^n)&-(-4)^n\\-(2^n)&(-4)^n\end{pmatrix}\begin{pmatrix}1&1\\1&3\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}3(2^n)-(-4)^n&3(2^n)-3(-4)^n\\-(2^n)+(-4)^n&-(2^n)+3(-4)^n\end{pmatrix}.\end{aligned}$$

(b) From the given information that $\mathbf{A}=\mathbf{PDP}^{-1}$, we can tell (from $\mathbf{D}$ and $\mathbf{P}$) that $\mathbf{A}$ has eigenvalues 2 and -4 with corresponding eigenvectors $\begin{pmatrix}3\\-1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$, respectively.

The eigenvectors tell us that the corresponding eigenlines are $y=-\frac{1}{3}x$ and $y=-x$, respectively.

(c) (i) The initial point $(-3,1)$ lies on the eigenline $y=-\frac{1}{3}x$, with eigenvalue 2. Hence from the table on page 41 of Chapter B3, the iteration sequence moves away from the origin, remaining on the same half of the line as $(-3,1)$.

(ii) The initial point $(0,0)$ is fixed by the linear transformation corresponding to $\mathbf{A}$, so the iteration sequence is the constant sequence $(x_n,y_n)=(0,0)$, for $n=0,1,2,\ldots$.

(iii) The initial point $(4,-2)$ is not on either eigenline. We use the iteration properties of generalised scalings.

By iteration property (a): since $2>0$, the points of the sequence (x_n,y_n) all lie on the same side of the eigenline $y=-x$; since $-4<0$, the points alternate between opposite sides of the eigenline $y=-\frac{1}{3}x$.

By iteration property (b): since $\max\{|2|,|-4|\}=4>1$, the sequence moves away from $(0,0)$.

Since $|-4|>|2|$, the dominant eigenvalue is -4 and the dominant eigenline is $y=-x$. Hence by iteration property (c) (the Dominant Eigenvalue Property)

$$\frac{y_n}{x_n}\to -1 \text{ as } n\to\infty.$$

Thus the sequence tends in the direction of the line $y=-x$, moving away from $(0,0)$. It stays on the same side of that line as the initial point, the points alternating between opposite sides of the line $y=-\frac{1}{3}x$.

(iv) The initial point $(-3,3)$ lies on the eigenline $y=-x$, with eigenvalue -4. Hence from the table on page 41 of Chapter B3, the iteration sequence moves away from $(0,0)$, alternating between the halves of the line $y=-x$.

Solution 5.1

(a) The initial point $(-6,3)$ lies on the eigenline $y=-\frac{1}{2}x$, with eigenvalue $\frac{1}{2}$. Hence from the table on page 41 of Chapter B3, the iteration sequence moves towards $(0,0)$, on the same half of the line $y=-\frac{1}{2}x$ as the initial point.

(b) The initial point $(5,4)$ is not on either eigenline. We use the iteration properties of generalised scalings.

By iteration property (a): since $\frac{1}{2}>0$, the points of the sequence (x_n,y_n) all lie on the same side of the eigenline $y=-\frac{2}{3}x$ as the initial point; since $-\frac{1}{3}<0$, the points alternate between opposite sides of the eigenline $y=-\frac{1}{2}x$.

By iteration property (b): since $\max\{|\frac{1}{2}|, |-\frac{1}{3}|\} = \frac{1}{2} < 1$, the sequence moves towards $(0,0)$.

Since $|\frac{1}{2}| > |-\frac{1}{3}|$, the dominant eigenvalue is $\frac{1}{2}$ and the dominant eigenline is $y = -\frac{1}{2}x$. Hence by iteration property (c) (the Dominant Eigenvalue Property)

$$\frac{y_n}{x_n} \to -\tfrac{1}{2} \text{ as } n \to \infty.$$

Thus the sequence tends in the direction of the line $y = -\frac{1}{2}x$, moving towards $(0,0)$. The points alternate between opposite sides of that line and stay on the same side of the line $y = -\frac{2}{3}x$ as the initial point.

Solution 5.2

(a) From the given information that $\mathbf{A} = \mathbf{PDP}^{-1}$, we can tell (from $\mathbf{D}$ and $\mathbf{P}$) that $\mathbf{A}$ has eigenvalues 2 and $-\frac{1}{2}$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, respectively. The eigenvectors tell us that the corresponding eigenlines are $y = 2x$ and $y = 4x$, respectively.

(b) (i) The initial point $(3, 12)$ lies on the eigenline $y = 4x$, with eigenvalue $-\frac{1}{2}$. Hence from the table on page 41 of Chapter B3, the iteration sequence moves towards $(0,0)$, alternating between the halves of the line $y = 4x$.

(ii) The initial point $(2, -2)$ is not on either eigenline. We use the iteration properties of generalised scalings.

By iteration property (a): since $2 > 0$, the points of the sequence (x_n, y_n) all lie on the same side of the eigenline $y = 4x$ as the initial point; since $-\frac{1}{2} < 0$, the points alternate between opposite sides of the eigenline $y = 2x$.

By iteration property (b): since $\max\{|2|, |-\frac{1}{2}|\} = 2 > 1$, the sequence moves away from $(0,0)$.

Since $|2| > |-\frac{1}{2}|$, the dominant eigenvalue is 2 and the dominant eigenline is $y = 2x$. Hence by iteration property (c) (the Dominant Eigenvalue Property)

$$\frac{y_n}{x_n} \to 2 \text{ as } n \to \infty.$$

Thus the sequence tends in the direction of the line $y = 2x$, moving away from the origin. The points alternate between opposite sides of that line and stay on the same side of the line $y = 4x$ as the initial point.

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Exercise Book B

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